Quantitative Isoperimetry à la Levy-Gromov

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(joint work with Francesco Maggi, Andrea Mondino)

Comparison theorems are an important part of Riemannian Geometry. The typical result asserts that a complete Riemannian manifold with a pointwise curvature bound retains some metric properties of the corresponding simply connected model space. We are interested here in the *Levy-Gromov comparison Theorem*, stating that, under a positive lower bound on the Ricci tensor, the isoperimetric profile of the manifold is bounded from below by the isoperimetric profile of the sphere. More precisely, define the isoperimetric profile of a smooth Riemannian manifold (M, g) by

$$\mathcal{I}_{(M,g)}(v) = \inf \left\{ \frac{\mathsf{P}(E)}{\mathrm{vol}_g(M)} : \frac{\mathrm{vol}_g(E)}{\mathrm{vol}_g(M)} = v \right\} \qquad 0 < v < 1\,,$$

where $\mathsf{P}(E)$ denotes the perimeter of a region $E \subset M$. The Levy-Gromov comparison Theorem states that, if $\mathrm{Ric}_g \geq (N-1)g$, where N is the dimension of (M, g), then

$$\mathcal{I}_{(M,g)}(v) \geq \mathcal{I}_{(\mathbb{S}^N, g_{\mathbb{S}^N})}(v) \qquad \forall v \in (0,1)\,, \tag{1}$$

where $g_{\mathbb{S}^N}$ is the round metric on $\mathbb{S}^N$ with unit sectional curvature; moreover, if equality holds in (1) for some $v \in (0, 1)$, then $(M, g) \simeq (\mathbb{S}^N, g_{\mathbb{S}^N})$.

Our main result is a quantitative estimate, in terms of the gap in the Levy-Gromov inequality, on the shape of isoperimetric sets in (M, g). We show that isoperimetric sets are close to geodesic balls. Since the classes of isoperimetric sets and geodesic balls *coincide* in the model space $(\mathbb{S}^N, g_{\mathbb{S}^N})$, one can see of our main result as a *quantitative comparison theorem*. In detail, we show that if $\mathrm{Ric}_g \geq (N-1)g$ and $E \subset M$ is an isoperimetric set in M with $\mathrm{vol}_g(E) = v\,\mathrm{vol}_g(M)$, then there exists $x \in M$ such that

$$\frac{\mathrm{vol}_g\big(E \Delta B_{r_N(v)}(x)\big)}{\mathrm{vol}_g(M)} \leq C(N, v) \Big(\mathcal{I}_{(M,g)}(v) - \mathcal{I}_{(\mathbb{S}^N, g_{\mathbb{S}^N})}(v) \Big)^{\mathrm{O}(1/N)} \tag{2}$$

where $B_r(x)$ denotes the geodesic ball in (M, g) with radius r and center x, and where $r_N(v)$ is the radius of a geodesic ball in $\mathbb{S}^N$ with volume $v\,\mathrm{vol}_{g_{\mathbb{S}^N}}(\mathbb{S}^N)$. More generally the same conclusion holds for every $E \subset M$ with $\mathrm{vol}_g(E) = v\,\mathrm{vol}_g(M)$, provided $\mathcal{I}_{(M,g)}(v)$ on the right-hand side of (2) is replaced by $\mathsf{P}(E)/\mathrm{vol}_g(M)$. In the course of proving (2), we improve on another basic comparison result, namely, *Meyer's Theorem*: if $\mathrm{Ric}_g \geq (N-1)g$, then $\mathrm{diam}(M) \leq \pi$. Indeed, we prove that

$$\pi - \mathrm{diam}(M) \leq \inf_{v \in (0,1)} C(N, v) \Big(\mathcal{I}_{(M,g)}(v) - \mathcal{I}_{(\mathbb{S}^N, g_{\mathbb{S}^N})}(v) \Big)^{1/N}. \tag{3}$$

We approach the proof of (2) and (3) from the synthetic point of view of metric geometry. We regard an N-dimensional Riemannian manifold (M, g) with $\mathrm{Ric}_g \geq (N-1)g$ as a metric measure space $(X, \mathsf{d}, \mathfrak{m})$ satisfying the curvature-dimension condition $\mathsf{CD}(N-1, N)$ of Sturm [3, 4] and Lott–Villani [2] and we actually obtain

(2) in this larger class. It is worth to underline that indeed in [1] the Levy-Gromov comparison Theorem (1) has been proved to hold for on essentially non-branching metric measure spaces verifying the $\mathsf{CD}(N-1,N)$ condition with *any real number* $N>1$.

References

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Optimal isoperimetric inequalities for surfaces in Cartan-Hadamard manifolds via mean curvature flow

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The classical isoperimetric inequality in Euclidean space states that for a bounded open set $\Omega \subset \mathbb{R}^{n+1}$ with sufficiently regular boundary, the estimate

$$|\Omega| \leq n^{-\frac{n+1}{n}}\omega_{n+1}^{-\frac{1}{n}}|\partial\Omega|^{\frac{n+1}{n}} \tag{1}$$

holds. Here $|\Omega|$ and $|\partial\Omega|$ denote the areas in dimension $n+1$ and n respectively. ω_{n+1} is the measure of the $n+1$-dimensional unit ball. Equality is attained if and only if Ω is a ball.

This was extended to higher codimension by Almgren; loosely formulated as follows:

Theorem 1 (Almgren, [1]**)**: *Corresponding to each m-dimensional closed surface T in $\mathbb{R}^{n+1}$ there is an $(m+1)$-dimensional surface Q having T as boundary such that*

$$|Q| \leq \gamma_{m+1}|T|^{\frac{m+1}{m}}$$

with equality if and only if T is a standard round m sphere (of some radius) and Q is the corresponding flat disk.

Again $|Q|$ and $|T|$ denote the areas in dimensions $m+1$ and m respectively, and the constant γ_{m+1} is defined via the required equality.

The proof of this inequality is based on the following area-mean curvature characterisation of standard spheres by Almgren:

Theorem 2 (Almgren, [1]**)**: *Let V be a (sufficiently regular) m-dimensional surface in $\mathbb{R}^{n+1}$ without boundary. Then the following estimate holds*

$$m^m|S_1^m| \leq (\sup_V|\vec{H}|)^m|V|\,,$$

where $\vec{H}$ is the mean curvature vector of V and S_1^m the standard unit round m-dimensional sphere. Furthermore, equality holds if and only if V is a standard