Plateau's problem as a singular limit of capillarity problems

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Minimal surfaces with prescribed boundary provide the basic model for soap films hanging from a wire frame: given a boundaryless $(n-1)$-dimensional surface $\Gamma \subset \mathbb{R}^{n+1}$, one looks for n-dimensional surfaces M with $\partial M = \Gamma$ and with zero mean curvature $H_M = 0$; here $n \geq 1$, with $n = 2$ in the physical case. A limitation in the descriptive power of this model is that it has no length scale (e.g., $t\,M$ is a minimal surface with boundary $t\,\Gamma$, no matter how large $t > 0$ is). Since the most basic length scale involved in this problem is the volume ε of the soap film, we are led to the question of framing Plateau's problem in the context of capillarity theory (area minimization at fixed volume).

As a reference formulation of Plateau's problem we adopt the one introduced by Harrison and Pugh in [1], in its slight generalization considered in [2],

$$\ell = \inf\Big\{\mathcal{H}^n(S) : S \text{ relatively closed in } \Omega,\ S \text{ is } \mathcal{C}\text{-spanning } W\Big\}. \tag{1}$$

Here W is the region occupied by the wire (think of a δ-neighborhood of Γ for a small $\delta > 0$), $\Omega = \mathbb{R}^{n+1} \setminus W$ is the region accessible to the candidate surfaces, $\mathcal{C}$ is a homotopically non-trivial and homotopically closed class of smooth embeddings of $\mathbb{S}^1$ into Ω, and S is said to be $\mathcal{C}$-spanning W iff $S \cap \gamma \neq \emptyset$ for every $\gamma \in \mathcal{C}$. By the results in [1, 2], minimizers of (1) exist as soon as $\ell < \infty$. Moreover, they satisfy Plateau's laws in the physical case $n = 2$ by [5].

The capillarity problem we want to consider is thus

$$\psi(\varepsilon) = \inf\Big\{\mathcal{H}^n(\Omega \cap \partial E) : |E| = \varepsilon,\ E \subset \Omega,\ \Omega \cap \partial E \text{ is } \mathcal{C}\text{-spanning } W\Big\}, \tag{2}$$

where the sets E are assumed to be open and such that ∂E is $\mathcal{H}^n$-rectifiable. We notice that the spanning condition on $\Omega \cap \partial E$ is imposed to exclude that global minimizers look like round droplets sitting a points of high curvature of ∂W, and to force them to actually resemble soap films. Given a minimizing sequence $\{E_j\}_j$ converging to a limit set E, an obvious difficulty is that $\Omega \cap \partial E_j$ may converge with multiplicity larger than 1 towards a surface K which strictly contains ∂E. Denoting by $\partial^* E$ the reduced boundary of a set of finite perimeter, we first prove the following existence result:

Theorem 1. *Assume that $\ell < \infty$, that ∂W is smooth, that there exists $\tau_0 > 0$ such that $\mathbb{R}^{n+1} \setminus I_\tau(W)$ is connected for every $\tau < \tau_0$, and that there exists a minimizer S of ℓ and $\eta_0 > 0$ such that $I_{\eta_0}(S) \cap \gamma \neq \emptyset$ for every $\gamma \in \mathcal{C}$.*

If $\{E_j\}_j$ is minimizing sequence for $\psi(\varepsilon)$, then, up to possibly extracting a subsequence and up to possibly modify each E_j outside of a large ball containing W (where both operations are still defining a minimizing sequence, which, for simplicity, is still denoted by $\{E_j\}_j$), we have

$$E_j \to E \text{ in } L^1(\mathbb{R}^{n+1}), \tag{3}$$

$$\mathcal{H}^n \llcorner (\Omega \cap \partial E_j) \overset{*}{\rightharpoonup} 2\,\mathcal{H}^n \llcorner (K \setminus \partial^* E) + \mathcal{H}^n \llcorner \partial^* E,\ \text{as Radon measures}, \tag{4}$$

as $j \to \infty$, where $E \subset \Omega$ is an open set with $\Omega \cap \partial E = \Omega \cap \mathrm{closure}(\partial^* E)$, $|E| = \varepsilon$, $\Omega \cap \partial E \subset K$, and K is an $\mathcal{H}^n$-rectifiable and relatively compact subset of Ω such that K is $\mathcal{C}$-spanning W. Moreover, $\mathcal{F}(K,E) = \psi(\varepsilon)$, where

$$\mathcal{F}(K,E) = 2\,\mathcal{H}^n(K \setminus \partial^* E) + \mathcal{H}^n(\partial^* E)\,,$$

is the relaxed surface tension energy of (K,E).

Because of the identity $\mathcal{F}(K,E) = \psi(\varepsilon)$, a pair (K,E) as in Theorem 1 is called a *generalized minimizer of* $\psi(\varepsilon)$. When $K = \Omega \cap \partial E$, then E is a minimizer of $\psi(\varepsilon)$, but in general K could be strictly larger than $\Omega \cap \partial E$, and in the latter situation we say that *collapsing* occurs.

When W consists of two small disjoint disks in the plane, then $K = \Omega \cap \partial E$ consists of two very flat circular arcs touching W orthogonally. In this case the region E has indeed a small thickness, proportional to ε. At fixed ε, by moving the two disks far away we see that this thickness becomes increasingly smaller. Below a certain thickness threshold, punctured configurations becomes energetically very close to the minimizer, and the probability transition towards such unstable states becomes increasingly consistent. This is an example of a physical feature of actual soap films which is unaccessible to a formulation via minimal surfaces.

When W is obtained by taking a δ-neighborhood of the three vertexes of an equilateral triangle, then the unique minimizer in ℓ consists of a triple junction at the center of the triangle. For small ε, collapsing occurs in $\psi(\varepsilon)$, and the generalized minimizer (K,E) consists of a central circular curvilinear triangle of area ε, joined to the boundary W by three segments of multiplicity 2. This result indicates that, in the presence of singularities, the volume and the thickness of an actual soap film are independent physical parameters. This is another physical feature of soap films which cannot be described relying just on minimal surfaces.

Theorem 2. *Under the assumptions of Theorem 1, if (K,E) is a generalized minimizer of $\psi(\varepsilon)$ and f is a diffeomorphism of Ω into Ω such that $|f(E)| = \varepsilon$, then $\mathcal{F}(K,E) \le \mathcal{F}(f(K), f(E))$. In particular there exists $\lambda \in \mathbb{R}$ such that*

$$\lambda \int_{\partial^* E} X \cdot \nu_E \, d\mathcal{H}^n = \int_{\partial^* E} \mathrm{div}^T X \, d\mathcal{H}^n + 2 \int_{K \setminus \partial^* E} \mathrm{div}^T X \, d\mathcal{H}^n$$

whenever $X \in C_c^\infty(\mathbb{R}^{n+1};\mathbb{R}^{n+1})$ with $X \cdot \nu_\Omega = 0$ along $\partial\Omega$, and where div^T denotes the tangential divergence operator.

Thanks to Theorem 2 and to Allard's regularity theorem for integer rectifiable varifolds, we find the existence of a closed set $\Sigma \subset K$, relatively meager in K, such that $K \setminus \Sigma$ is a smooth hypersurface. In fact, $K \setminus (\Sigma \cup \partial E)$ is a smooth minimal surface, $\partial^* E$ is a smooth hypersurface with constant mean curvature equal to λ, $\mathcal{H}^n(\Sigma \setminus \partial E) = 0$ and $\partial E \setminus \partial^* E$ is meager in K and contained in Σ. We thus have two interesting free boundary problems: (i) on the transition region $\partial E \setminus \partial^* E$; (ii) on the wetted region of the wire $\partial W \cap \mathrm{closure}(K \cup E)$.

Finally, we discuss the convergence towards Plateau's problem.

Theorem 3. *Under the assumptions of Theorem 1, $\psi(\varepsilon) \to 2\,\ell$ as $\varepsilon \to 0^+$. More precisely, if (K_j, E_j) is a sequence of generalized minimizers of $\psi(\varepsilon_j)$ for some $\varepsilon_j \to 0^+$ as $j \to \infty$, then, up to possibly extracting a subsequence, there exists a minimizer S of ℓ such that, as $j \to \infty$,*

$$2\,\mathcal{H}^n \llcorner (K_j \setminus \partial^* E_j) + \mathcal{H}^n \llcorner \partial^* E_j \overset{*}{\rightharpoonup} 2\,\mathcal{H}^n \llcorner S, \quad \text{as Radon measures.} \tag{5}$$

Thus Plateau's problem ℓ is the singular limit as $\varepsilon \to 0^+$ of the capillarity problems $\psi(\varepsilon)$, and this limit provides a selection principle for minimizers in ℓ based on the size of their singular sets. For example, in the planar case, simple examples show that generalized minimizers of $\psi(\varepsilon)$ will necessarily converge to those minimizers of ℓ with the largest number of singular points.

For a more complete discussion on the physical and mathematical meaning of this singular limit we refer to the two papers [4, 3]. In particular, the three theorems above are proved in [3].

References

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