ALMOST-CRITICAL POINTS IN GEOMETRIC VARIATIONAL PROBLEMS

FRANCESCO MAGGI

ICTP TRIESTE

James Serrin : from his legacy to the new frontiers

BOUNDARIES WITH ALMOST CONSTANT MEAN CURVATURE

MOTIVATIONS: CAPILLARITY THEORY
EXTRINSIC CURVATURE FLOWS
CMC FOLIATIONS IN ASYMPT. EUCLIDEAN MANIFOLDS

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NOTATION H_Ω SCALAR MEAN CURV. W.R.T. ν_Ω EXT. 1 NORMAL

$\Omega \subseteq \mathbb{R}^{n+1}$, $H_{B_1} = n$

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DEF. CMC DEFICIT

$$\delta_{CMC}(\Omega) = \left\| \frac{H_\Omega}{H^0_\Omega} - 1 \right\|_{C^0(\partial\Omega)}$$

RECALL IF $H_\Omega \equiv \lambda$ THEN $\lambda = H^0_\Omega = \dfrac{n\,P(\Omega)}{(n+1)\,|\Omega|}$

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QUESTION HOW THE SMALNESS OF $\delta_{cmc}(\Omega)$ CONTROLS THE SHAPE OF Ω?

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WHY C^0? CAPILLARITY THEORY

CAPILLARITY FUNCTIONAL

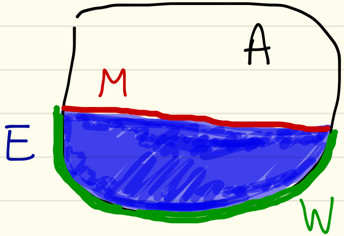
A CONTAINER

M LIQUID/AIR INTERFACE

W WETTED SURFACE

E DROPLET

$\sigma : \partial A \to (-1,1)$ RELATIVE ADHESION COEFFICIENT



$$\boxed{\mathcal{F}(E) = \underbrace{\text{Area}(M) + \int_W \sigma}_{\text{SURFACE TENSION}} + \underbrace{\int_E g(x)\,dx}_{\text{POTENTIAL ENERGY}}}$$

VOLUME CONSTRAINT $|E| = m$

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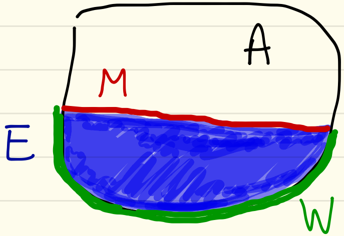

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VOLUME CONSTRAINT $|E| = m$

CAPILLARITY: $\underbrace{\text{SURFACE TENSION}}_{O(m^{n-1/n})} \gg \underbrace{\text{POTENTIAL ENERGY}}_{O(m)}$

GOAL: UNDERSTANDING GLOBAL/LOCAL MINIMIZERS

& CRITICAL POINTS IN THE SMALL VOLUME REGIME

GLOBAL MINIMIZERS IN $\mathbb{R}^n$ (NO CONTAINER)

$$\boxed{\begin{array}{l} P(E) + \int_E g \leq P(F) + \int_F g \\ \text{FOR EVERY } |F| = |E| = m \end{array}}$$

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$$P(E)-P(B) \le \int_B g - \int_E g \le C\,|E \Delta B|$$
$$= O(m)$$

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$$\underbrace{c\,P(B)\left(\frac{|E\Delta(x+B)|}{|E|}\right)^2}_{\text{SHARP QUANTITATIVE ISOPERIMETRIC INQ}} \le P(E)-P(B) \le \int_B g-\int_E g \le C\,|E\Delta B| = O(m)$$

SHARP QUANTITATIVE

ISOPERIMETRIC INQ

FUSCO M. PRATELLI ANN M. 08

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SHARP QUANTITATIVE

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AS $|E\Delta B| \le 2m \qquad P(B)=m^{(n-1)/n}$

WE HAVE

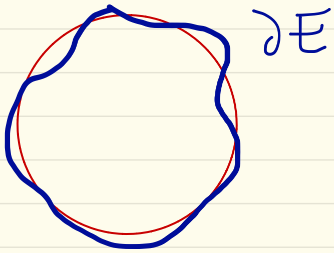
$$\boxed{\frac{|E\Delta(x+B)|}{|E|} \le C\,m^{1/2n}}$$

THM FIGALLI M. ARMA 11

IF E GLOBAL MINIMIZER OF $P(E)+\int_E g$ WITH $g\in C^1_{loc}(\mathbb{R}^n)$

& g COERCIVE THEN $|E|=m<m_0(n,g)$ IMPLIES

$$\frac{R^{OUT}(E)}{R^{IN}(E)}\leq 1+C\,m^{1/n^2}$$



$$\left\| m^{1/n}k_i^{\partial E}-c(n)\right\|_{C^0(\partial E)}\leq C\,m^{\frac{2}{n+2}}$$

IN PARTICULAR E IS CONVEX

PHYSICAL MOTIVATION

$\Longrightarrow$ LOCAL MINIMIZERS / CRITICAL POINTS

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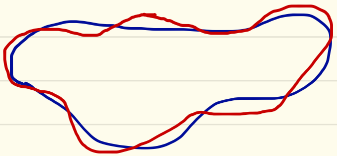

whenever $F \Delta E \subset\subset \underbrace{I_\varepsilon(\partial E)}$, $|F| = |E|$

ε NEIGHBORHOOD

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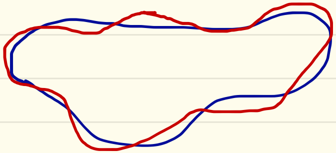

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CRITICAL POINTS $\exists\, \lambda \in \mathbb{R}$ S.T. $H_E + g = \lambda$ ON ∂E

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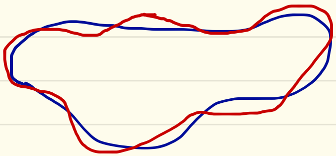

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RMK WHEN $g = 0$ THEN H_E IS CONSTANT

WHEN $g \neq 0$ & m SMALL, THEN H_E IS ALMOST-CONSTANT

WHY ALMOST CMC:

$$H_E + g = \lambda \text{ ON } \partial E \implies \lambda = H_E^{0} + \underbrace{\frac{1}{|E|}\int_E \operatorname{div}(g(x)x)\,dx}_{} = O(1)$$

$g = O(1)$ IF E BOUNDED

$$H_E^{0} = \frac{n\,P(E)}{(n+1)\,|E|} = O\left(\frac{m^{\frac{n}{n+1}}}{m}\right)$$

$$= O\left(m^{-1/n+1}\right)$$

$$\implies \frac{H_E - H_E^{0}}{H_E^{0}} = \frac{O(1)}{O(m^{-1/n+1})} = O(m^{1/n+1})$$

$$\implies \delta_{cmc}(E) = \left\| \frac{H_E}{H_E^{0}} - 1 \right\|_{C^0(\partial E)} = O(m^{1/n+1}) \text{ AS } m \to 0^+$$

SETS WITH SMALL $\delta_{cmc}(\Omega)$

WITH CONVEXITY HYP.: SCHNEIDER 90 KOHLMANN 00 ARNOLD 93

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QUANTITATIVE MOVING PLANES METHOD

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QUANTITATIVE MOVING PLANES METHOD

THM (CIRAOLO-VEZZONI 2015)

IF $\exists\, \rho > 0$ SUCH THAT Ω HAS EXT/INT BALL CONDITION

WITH RADIUS ρ & $\delta_{cmc}(\Omega) \leq \delta_0(n, P(\Omega)/\rho^n)$ THEN

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HAUSDORFF DISTANCE

SETS WITH SMALL $\delta_{cmc}(\Omega)$

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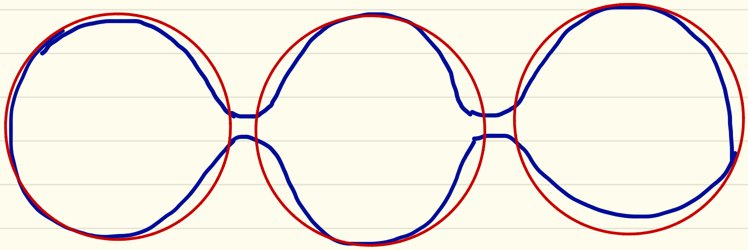
$$\underbrace{hd}_{\text{HAUSDORFF DISTANCE}}(\partial\Omega, \text{A SPHERE}) \leq C_0\, \delta_{cmc}(\Omega)$$

SHARP ESTIMATE

ρ NOT NATURAL

SLIGHTLY PERTURBED UNDULOIDS



THM CIRAOLO-M CPAM 17

E OPEN BOUNDED CLASS C^2

$$H_E^0 = \frac{nP(E)}{(n+1)|E|} = n \quad \text{(SCALING)}$$

$$\boxed{\begin{array}{l} H_E^0 = \dfrac{nP(E)}{(n+1)|E|} = n \quad \text{(SCALING)} \\ P(E) \leq (L+1-a)P(B_1) \quad \text{SOME } L \in \mathbb{N},\ a \in (0,1) \\ \delta(E) \leq \delta_0(n, L, a) \end{array}}$$

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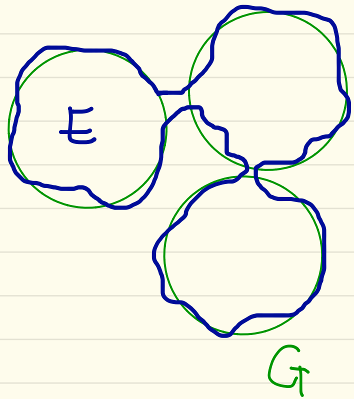
THEN $\exists \{B_1(z_j)\}_{j=1}^N$ DISJOINT UNIT BALLS

$$G = \bigcup_{j=1}^N B_1(z_j) \qquad N \le L$$



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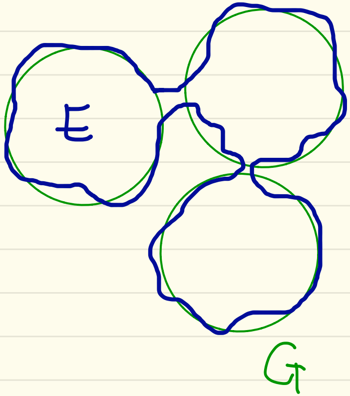
$$\frac{|E \Delta G|}{|E|} + \frac{|P(E) - NP(B_1)|}{P(E)} \le C\,\delta(E)^{1/2(n+2)}$$



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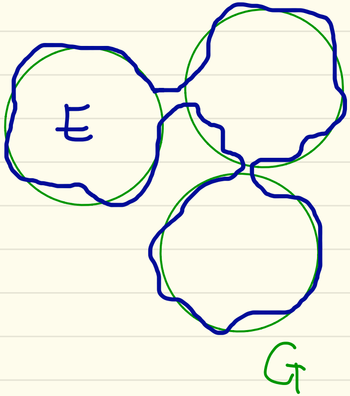
$$\frac{hd(\partial E, \partial G)}{\operatorname{diam}(E)} \leq C\,\delta(E)^{\alpha} \qquad \alpha = O(n^{-4})$$

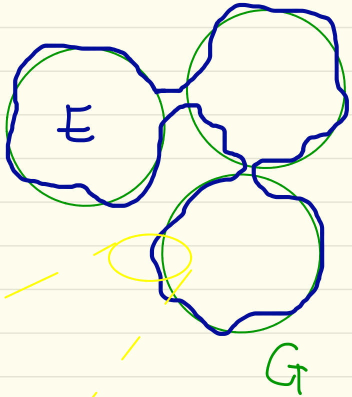


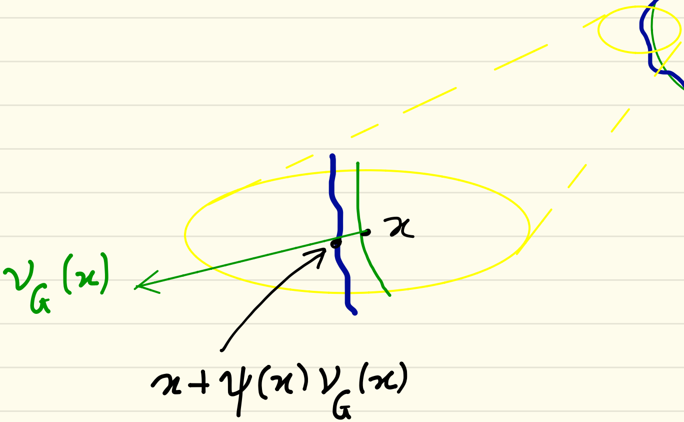
CONTINUED...

MOREOVER

$$\begin{cases} \exists\, \Sigma \subseteq \partial G \setminus \text{SPHERICAL CAPS W. DIAMETER} \leq \delta(E)^{\alpha} \\ \qquad\qquad \alpha = O(n^{-2}) \\ \exists\, \psi \in C^{1,\gamma}(\Sigma) \quad \forall \gamma \in (0,1) \end{cases}$$

S.T. $\quad (Id + \psi\, \nu_G)(\Sigma) \subseteq \partial E$





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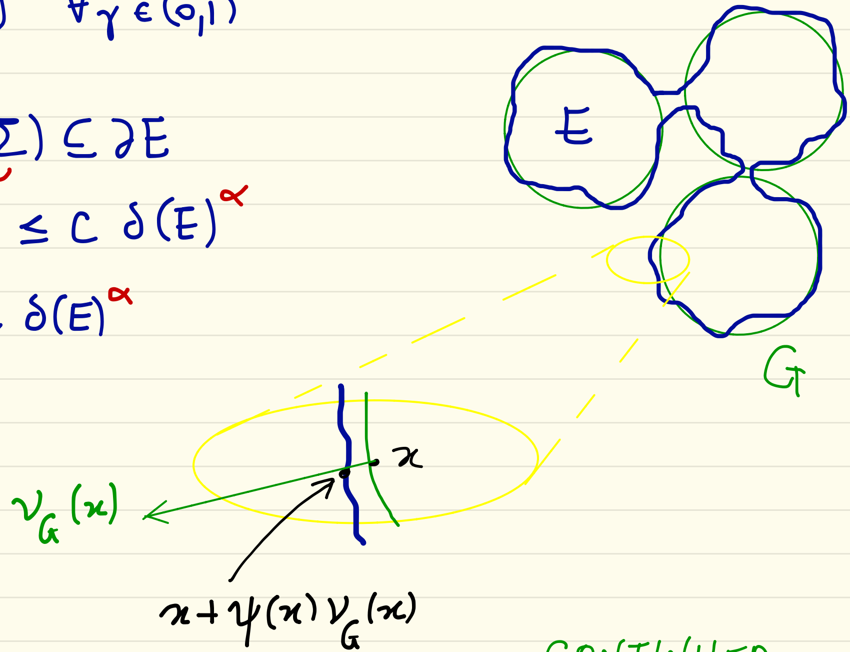
S.T.

$$\begin{cases} (Id + \psi \nu_G)(\Sigma) \subseteq \partial E \\ \mathcal{H}^n\big(\partial E \setminus (Id + \psi \nu_G)(\Sigma)\big) \le C\, \delta(E)^{\alpha} \\ \|\psi\|_{C^1(\partial E)} \le C\, \delta(E)^{\alpha} \\ \alpha = O(n^{-3}) \end{cases}$$



CONTINUED...

FINALLY 1 IF $N \geq 2$ THEN $\forall z_j \, \exists z_k$ SUCH THAT $\big| |z_j - z_k| - 2 \big| \leq C\, \delta(E)^{\alpha}$

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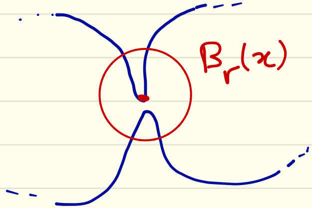
② IF $\exists K > 0$ S.T. $|E \setminus B_r(x)| \geq K |B_r(x)|$ $\forall x \in \partial E \; \forall r < K$

THEN $N = 1$



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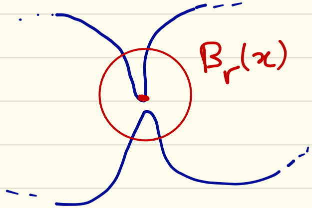
COROLLARY LOCAL MINIMIZERS WITH m SMALL

ARE CLOSE TO SINGLE SPHERES



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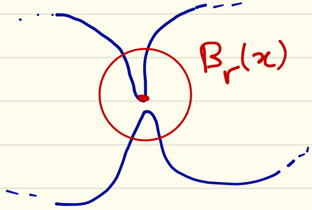

NOT SHARP

KRUMMEL-M. 16

$$\begin{cases} H^{\circ}_{\partial E} = n \qquad P(E) \leq (1+\tau)\, P(B) \qquad 0<\tau<1 \\ \delta_{CMC}(E) \leq \delta_0(n,\tau) \end{cases}$$

$$\Longrightarrow \quad \partial E = \{ (1+u(x))\, x : x \in \partial B_1 \} \qquad \|u\|_{C^{1,\alpha}} \leq C(n,\tau)\, \delta_{CMC}(E)$$

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IN ADDITION

$$\int_{\partial B_1} u^2 + |\nabla u|^2 \leq C(n,\tau) \int_{\partial E} |H_{\partial E} - n|^2$$

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RELATED TO ALMGREN ISOPERIMETRIC PRINCIPLE

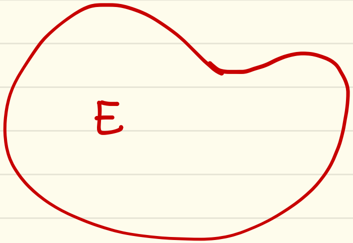
ALMGREN ISOPERIMETRIC PRINCIPLE

(CODIMENSION 1 VERSION)

IF $H_{\partial E} \leq n$ THEN $P(E) \geq P(B_1)$

ALMGREN ISOPERIMETRIC PRINCIPLE

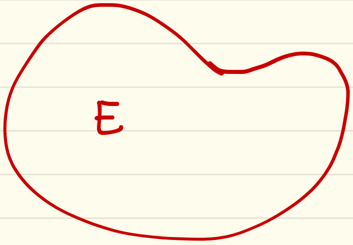
IF $H_{\partial E} \le n$ THEN $P(E) \ge P(B_1)$



$$P(B_1) = \mathcal{H}^n(S^n) = \int_{\partial A} |\det \nabla \nu_A|$$

ALMGREN ISOPERIMETRIC PRINCIPLE

IF $H_{\partial E} \leq n$ THEN $P(E) \geq P(B_1)$



$$P(B_1) = \mathcal{H}^n(S^n) = \int_{\partial A} |\det \nabla \nu_A|$$

$$= \int_{\partial A} K_{\partial A} = \int_{\partial A \cap \partial E} K_{\partial A}$$

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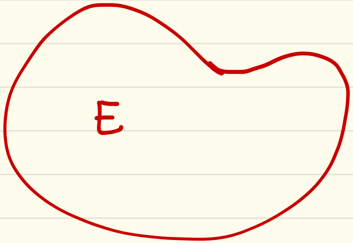

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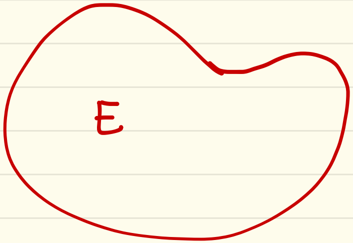

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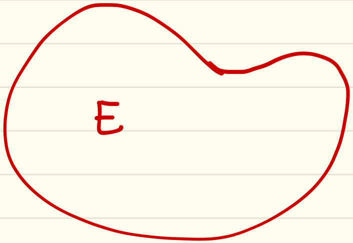

$$P(B_1) = \mathcal{H}^n(S^n) = \int_{\partial A} |D\nu_A|$$

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RMK 1 EQUALITY HOLDS

$\iff$ $E = B_1(x)$

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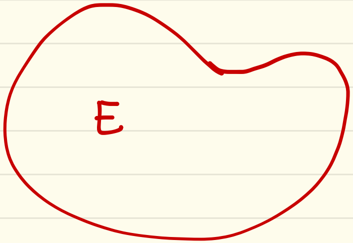

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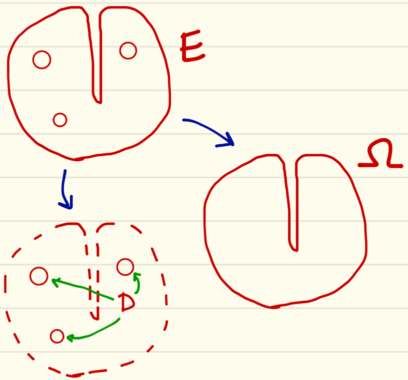
RMK 1 EQUALITY HOLDS

$\Longleftrightarrow \quad E = B_1(x)$

RMK 2 YES! IT REMINDS

A LOT ABP !

KRUMMEL-M. 16 IF $H_{\partial E} \leq n$ & $\delta(E) = P(E) - P(B_1) \leq \delta_0(n)$ SMALL



THEN $\partial E = \partial D \cup \partial \Omega$

D "DUST SET" $P(D) \leq C(n)\, \delta(E)$

$\partial \Omega$ CONNECTED

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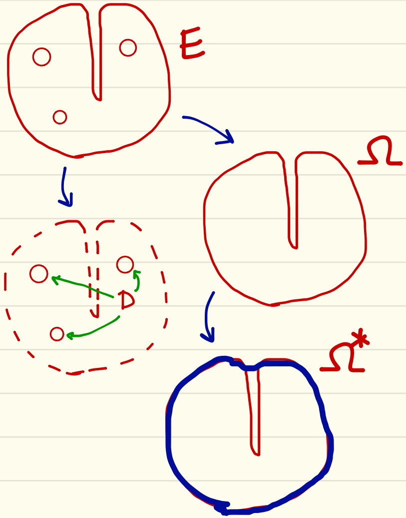

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$\Omega \subseteq \Omega^*$ $\quad |H_{\partial \Omega^*}| \leq n$

$$|\Omega^* \setminus \Omega| + \mathcal{H}^n(\partial \Omega^* \setminus \partial \Omega) \leq C(n)\, \delta(E)$$

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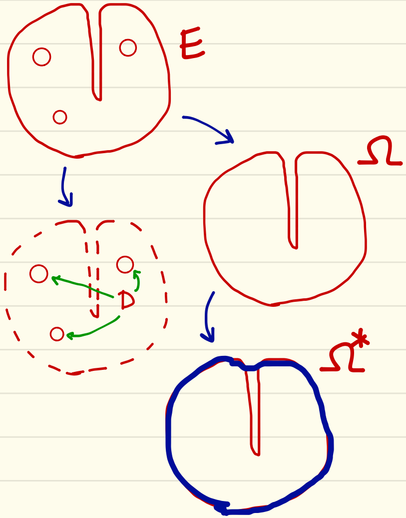

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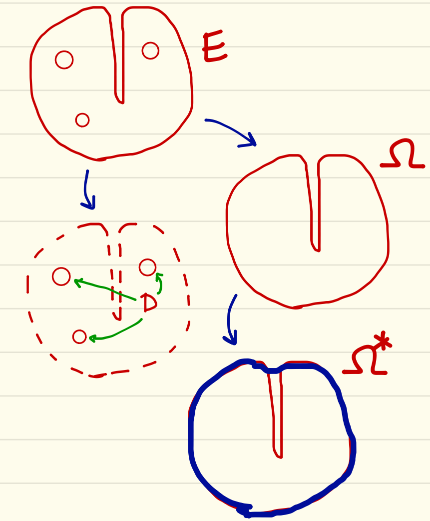
$$\partial\Omega^* = \{(1+u(x))\,x : x \in \partial B_1\}$$

$$\|u\|_{W^{1,1}} + \|u^+\|_{C^0} \le C(n)\,\delta(E) \qquad \|u\|_{C^0} \le C(n) \begin{cases} \delta(E) & n=1 \\ \delta(E)\log\left(C(2)/\delta(E)\right) & n=2 \\ \delta(E)^{1/n-1} & n \ge 3 \end{cases}$$

KRUMMEL-M. 16 IF $H_{\partial E} \le n$ & $\delta(E) = P(E) - P(B_1) \le \delta_0(n)$ SMALL



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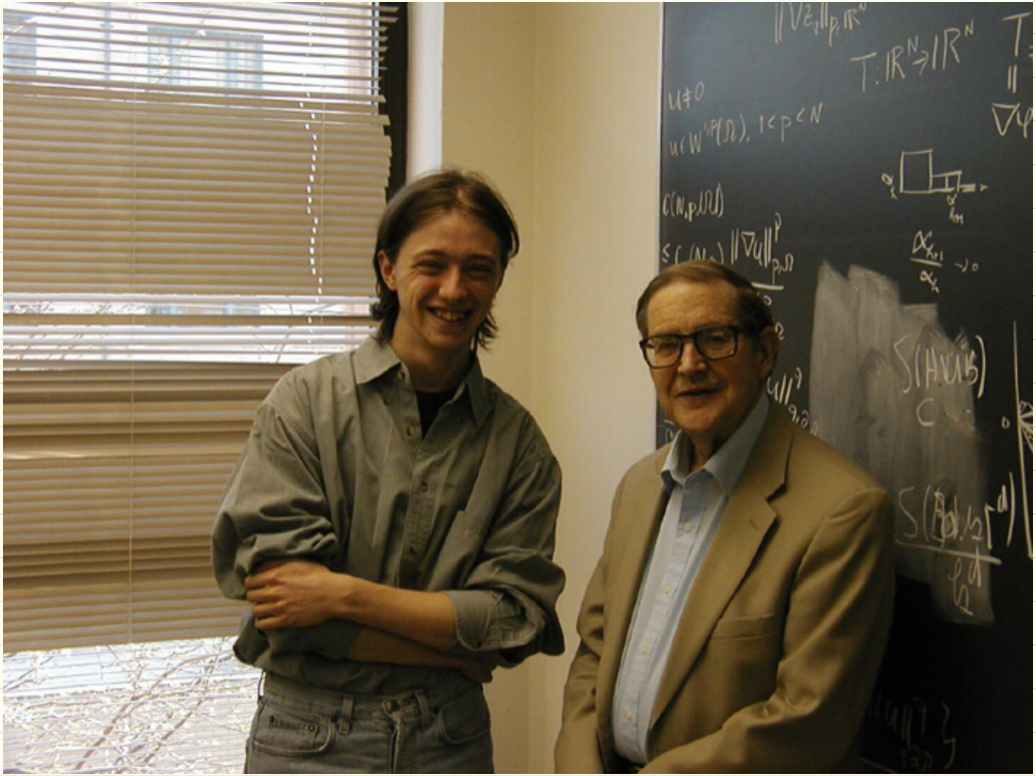
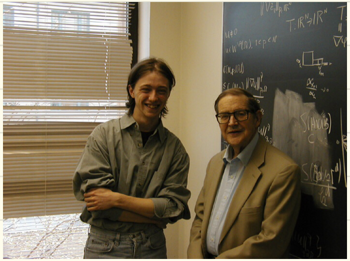
TRUNCATING MEAN CURVATURE
BY FREE BOUNDARY THEORY



T:IR^N→IR^N
u≠0
C(N,p,Ω)

SERRIN'S LOWER SEMICONTINUITY THM



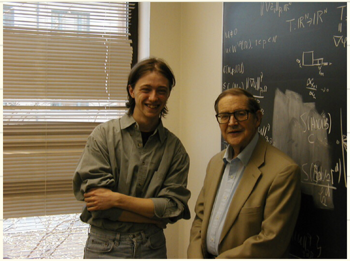
SUFFICIENT CONDITIONS FOR $\int_\Omega f(x,u(x),\nabla u(x))\,dx$ DEFINED ON $u:\Omega\to\mathbb{R}$ TO BE LSC ALONG EVERY $u_h\to u$ IN L^1_{loc} ARE:

EITHER ① $f(x,s,z)$ IS STRICTLY CONVEX IN z

OR ② $f(x,s,z)\to+\infty$ AS $|z|\to+\infty$ & f IS CONVEX IN z

OR ③ f is C^2 & f IS CONVEX IN z

SERRIN'S LOWER SEMICONTINUITY THM



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GORI-M. (02): ⓪ $f(x,s,z)$ CONVEX & NON-CONSTANT ON LINES IN z

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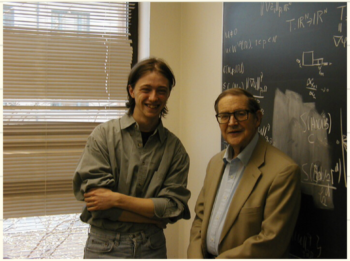

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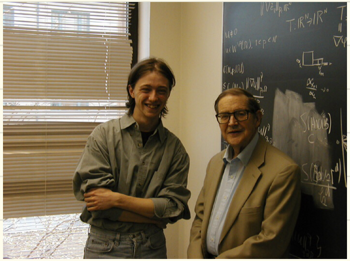
FUSCO-GORI-M (03): IF (0) HOLDS THEN $f = \sup_{k \in \mathbb{N}} f_k$ FOR f_k AS IN ③

SERRIN'S LOWER SEMICONTINUITY THM



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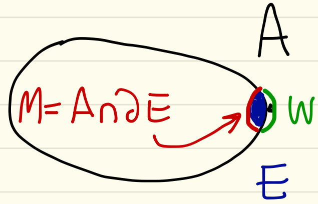
GORI-M.-MARCELLINI (02): ④ $f(x,s,z)$ CONVEX

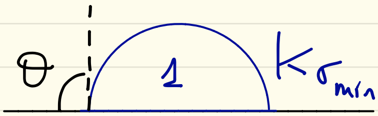
$f(\cdot, s, z) \in W^{1,1}_{loc}(\Omega)$

THM (MIHAILA, M. 15)

IF E GLOBAL MINIMIZER OF $\mathcal{H}^{n-1}(M)+\int_W \sigma+\int_E g$ WITH

$\partial A \in C^{1,1}$ $\sigma \in Lip(A)$ $g \in Lip(A)$





$\cos\theta = \sigma_{min}$

THM (MIHAILA, M. 15)

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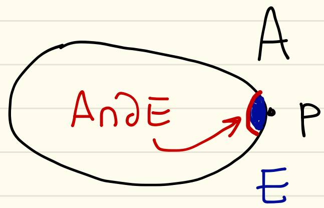
$\partial A \in C^{1,1}$ $\sigma \in Lip(A)$ $g \in Lip(A)$ THEN $|E| = m < m_0(n, g)$ IMPLIES

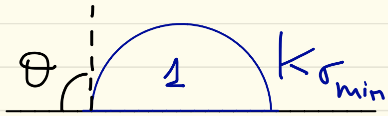
$\exists\, p \in \partial A$ & L ISOMETRY S.T.



$E \subseteq B_{cm^{1/n}}(p)$

$\sigma(p) - \sigma_{min} \leq C\, m^{1/n}$

$$hd\left(L\left(\frac{A \cap \partial E - p}{m^{1/n}} \right), K_\sigma \right) \leq C\, m^{1/2n^2}$$



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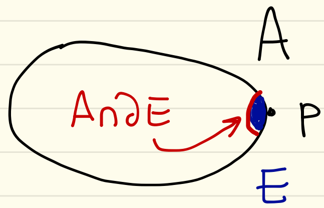
THM (MIHAILA, M. 15)

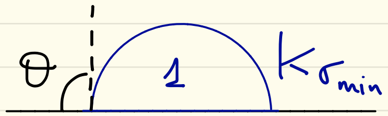
IF E GLOBAL MINIMIZER OF $\mathcal{H}^{n-1}(M)+\int_W \sigma+\int_E g$ WITH $\partial A\in C^{1,1}$ $\sigma\in Lip(A)$ $g\in Lip(A)$ THEN $|E|=m<m_0(n,g)$ IMPLIES $\exists\, p\in\partial A$ & L ISOMETRY S.T.



$E\subseteq B_{Cm^{1/n}}(p)$

$\sigma(p)-\sigma_{min}\leq C\, m^{1/n}$

$$hd\left(L\left(\frac{A\cap\partial E-p}{m^{1/n}}\right),K_\sigma\right)\leq C\, m^{1/2n^2}$$

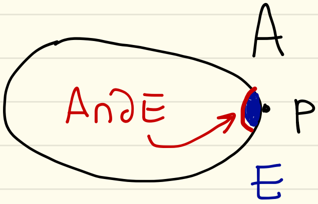


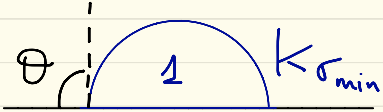
$\cos\theta=\sigma_{min}$

MOREOVER $A\cap\partial E$ IS $C^{1,\alpha}$ DIFFEO TO K_σ WITH $f\simeq p+m^{1/n}L$ & $ENERGY(E)=m^{\frac{n-1}{n}}C(\sigma_{min})\left(1+O(m^{1/n})\right)$

Moreover $A \cap \partial E$ is $C^{1,\alpha}$ diffeo to K_σ

with $f \approx p + m^{1/n} L$ & $\text{Energy}(E) = m^{\frac{n-1}{n}} c(\sigma_{min}) \left(1 + O(m^{1/n})\right)$





$\cos\theta = \sigma_{min}$

De Philippis–M. 14 Free boundary regularity

Cicalese–Leonardi–M. 17 Improved convergence