The Growth of Money

1. Interest

2. Accumulation and amount functions

3. Simple Interest/Linear Accumulation Functions

4. Discount functions/The time value of money

5. Simple discount
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What is interest?

- $K$ ... the principal, i.e., the amount of money that the **borrower** borrows/lender lends at time $t = 0$
- $S$ ... the amount of money that changes hands at a later time - say, $T$
- The interest is defined as

  $$S - K \geq 0$$

*Reading assignment: Section 1.2 in the textbook (on the rationale behind the existence of interest)*
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The amount function

- Let us temporarily fix the principal $K$
  - $A_K(t)$ ... the amount function for principal $K$, i.e., the balance at time $t \geq 0$ (time is always measured in some agreed upon units; think “years” for now)
    - In words: $K$ invested at time $t = 0$ “grows” to $A_K(t)$ at time $t \geq 0$
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The accumulation function

- \( a(t) \) ... the accumulation function, i.e., the amount function if the principal \( K \) is one dollar
- Formally: If the principal is one dollar, we write

  \[ a(t) = A_1(t) \]

- Note:

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The relationship between the amount and the accumulation functions

- We expect to have that

\[ A_K(t) = Ka(t) \]

- This is common, but is NOT always the case (the investment scheme may include a tiered growth structure).

- However, since the above equality holds in most cases, we will assume that it is true unless it is explicitly noted otherwise.
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The increase/decrease of the amount and the accumulation functions

- It is natural to assume that both $a$ and $A_K$ increase in the time variable.
- Such increase may be, for example:
  - continuous and linear;
  - discrete (end of the year, e.g.);
  - continuous and exponential
- However, there are investment schemes in which it is possible to lose money over time (e.g., if one invests in a fund that trades in the market or in a restaurant that takes time to pay off)
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The effective interest rate

Let \( t_2 > t_1 \geq 0 \)

- \( A_K(t_2) - A_K(t_1) \) \ldots the amount of interest earned between time \( t_1 \) and time \( t_2 \)
- \( i_{[t_1,t_2]} \) \ldots the effective interest rate for the interval \([t_1,t_2]\), i.e.,

\[
i_{[t_1,t_2]} = \frac{a(t_2) - a(t_1)}{a(t_1)}
\]

- If \( A_K(t) = Ka(t) \), then we also have

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- The interval \([n-1,n]\) is called the \( n^{th} \) time period (for \( n \) a positive integer)
- Notation:

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i_n = i_{[n-1,n]} = \frac{a(n) - a(n-1)}{a(n-1)}
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- Hence,

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a(n) = a(n-1)(1 + i_n)
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In this case, $a$ is assumed linear and, thus, must be of the form

$$a(t) = 1 + st$$

for a certain constant $s$

- $s$ ... the simple interest rate
- Note: $s = i_1$
- $A_K(t) = K(1 + st)$ ... the amount function for $K$ invested by simple interest at rate $s$
- $a(t) = 1 + st$ ... the simple interest accumulation function at rate $s$
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In the simple interest case:

\[ i_n = \frac{s}{1 + s(n-1)} \]

So, \( i_n \) is decreasing in \( n \) (see Example 1.4.2 in the textbook for an illustration of this fact).

Moreover,

\[ i_n \to 0, \text{ as } n \to \infty \]

So, simple interest is not convenient for long duration loans.
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Methods for measuring the time/length of the loan in years

• Exact simple interest aka "actual/actual"
  The loan term $D$ expressed in days and divided by 365

• Ordinary simple interest aka "30/360"
  The loan term $D$ expressed in days assuming that each month has 30 days and then divided by 360

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- $v(t)$...the **discount function**, i.e.,

$$v(t) = \frac{1}{a(t)}$$

- In words, $v(t)$ is the amount of money that one should **invest** at time 0 in order to have $1 at time $t$
- For example, in the simple interest case, we have that

$$v(t) = \frac{1}{1 + st}$$

- **Question:** What if one wishes to invest a certain amount not at time 0 but at a later time $t_1 > 0$ - with the goal of earning $S$ at a still later time $t_2$?
- Let us draw the time line ....
- One needs to invest (at time $t_1$)

$$Sv(t_2)a(t_1) = S \frac{a(t_1)}{a(t_2)} = S \frac{v(t_2)}{v(t_1)}$$

- **Assignment:** Examples 1.7.2, 1.7.3 in the textbook
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  \[ v(t) = \frac{1}{a(t)} \]

- In words, $v(t)$ is the amount of money that one should **invest** at time 0 in order to have $1 at time $t$
- For example, in the simple interest case, we have that
  \[ v(t) = \frac{1}{1 + st} \]

- **Question:** What if one wishes to invest a certain amount not at time 0 but at a later time $t_1 > 0$ - with the goal of earning $S$ at a still later time $t_2$?
- Let us draw the time line ....
- One needs to invest (at time $t_1$)
  \[ S v(t_2) a(t_1) = S \frac{a(t_1)}{a(t_2)} = S \frac{v(t_2)}{v(t_1)} \]

- **Assignment:** Examples 1.7.2, 1.7.3 in the textbook
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Present Value

- $PV_{a(t)}(L \text{ at } t_0)$ ... present value with respect to $a(t)$ of $L$ to be received at time $t_0$, i.e.,

$$PV_{a(t)}(L \text{ at } t_0) = L v(t_0)$$

if the growth is proportional to the invested amount

- Convention: If it is obvious which accumulation function $a(t)$ we use, we suppress it from the notation for the present value
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The Growth of Money

1. Interest
2. Accumulation and amount functions
3. Simple Interest/Linear Accumulation Functions
4. Discount functions/The time value of money
5. Simple discount
Discount rate

- $D \ldots$ the **discount** per unit time period, per dollar that the borrower and the lender agree upon at time 0, i.e.,

  If an investor (lender) lends $1$ for one basic period at a discount rate $D$ - this means that in order to obtain $1$ at time 0, the borrower must **pay immediately** $D$ to the lender.

- Note that the “net-effect” for the borrower is that they get to use $(1 - D)$ at time zero

- The initial fee is proportional to the amount of money borrowed, i.e., if one wants to borrow $K$, one needs to pay $DK$ to the lender

- The value $DK$ is called **the amount of discount**
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Simple discount

• $D$ ... the discount per unit time period, per dollar that the borrower and the lender agree upon at time 0, i.e., $D$ is uniquely determined by

$$a(t) = \frac{1}{1 - tD}$$

or, equivalently

$$v(t) = 1 - tD$$

• Note that the discount function $v(t)$ is linear in this case

• **Caveat:** This situation is not the same as the one when the accumulation function is linear.
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$A_K(t)$ and $a(t)$

- More vocabulary:

\[ A_K(t) = \frac{K}{1 - dt} \]

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- Let us draw their graphs .....

- Note that it only makes sense to talk about loan terms that are shorter than $1/d$
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