Annuities Governed by general accumulation functions

Review

- Let $a(t)$ denote the accumulation function which governs an annuity and let $v(t)$ denote the discount function, i.e., $v(t) = 1/a(t)$
- Then, in general

$$a_{\overline{n}|} = v(1) + v(2) + \cdots + v(n)$$
$$\ddot{a}_{\overline{n}|} = 1 + v(1) + v(2) + \cdots + v(n-1)$$

- Evidently,

$$\ddot{a}_{\overline{n}|} = 1 + a_{\overline{n-1}|}$$

- Multiplying the expressions for the values at issuance of the annuities above by $a(n)$, we obtain the expressions for their accumulated values at closing:

$$s_{\overline{n}|} = a(n) \cdot a_{\overline{n}|} = \frac{a(n)}{a(1)} + \frac{a(n)}{a(2)} + \cdots + 1$$
$$\ddot{s}_{\overline{n}|} = a(n) \cdot \ddot{a}_{\overline{n}|} = \frac{a(n)}{a(0)} + \frac{a(n)}{a(1)} + \cdots + \frac{a(n)}{a(n-1)}$$

- *Assignment:* See Example 3.12.4 in the textbook

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An Example: A constant simple discount rate d

- Find an expression for $a_{\overline{n|}}$ assuming each payment is valued at a simple discount rate d.

⇒ Recall that the discount function in the case of the simple discount rate has the form

$$v(n) = 1 - d \cdot n, \text{ for every } n \geq 0$$

Therefore,

$$\begin{aligned} a_{\overline{n|}} &= v(1) + v(2) + \cdots + v(n) \\ &= (1 - d \cdot 1) + (1 - d \cdot 2) + \cdots + (1 - d \cdot n) \\ &= n - d \cdot (1 + 2 + \ldots n) \\ &= n - d \cdot \frac{n(n+1)}{2} \end{aligned}$$

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An Example: Simple interest

- Let the accumulation function be given as

$$a(t) = 1 + 0.1t, \text{ for } t \geq 0$$

Find $s_{\overline{6}|}$.

$\Rightarrow$

$$s_{\overline{6}|} = 1.6 \left[\frac{1}{1.1} + \frac{1}{1.2} + \cdots + \frac{1}{1.6} \right] = 7.23$$

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An Example: Time-varying force of interest

- Consider a basic 5−year annuity-immediate. Assume that the underlying force of interest is given by $\delta_t = 0.02t$, for $t \in [0, 5]$. Find the accumulated value of this annuity at time 5.

⇒ The accumulated value of all payments at the end of 5 years is

$$s_{\overline{5}|} = \sum_{k=1}^{5} \frac{a(5)}{a(k)} = \sum_{k=1}^{5} \frac{e^{\int_0^5 \delta_t \, dt}}{e^{\int_0^k \delta_t \, dt}} = \sum_{k=1}^{5} e^{\int_k^5 \delta_t \, dt}$$

$$= \sum_{k=1}^{5} e^{\int_k^5 0.02t \, dt} = \sum_{k=1}^{5} e^{0.01t^2]_k^5}$$

$$= e^{0.24} + e^{0.21} + e^{0.16} + e^{0.09} + 1 = 5.7726$$

- *Assignment:* Examples 3.12.8,9,10
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An Example

- Find the accumulated value of a 10−year annuity-immediate of \$100 per year if the effective interest rate is 5% for the first 6 years and 4% for the last 4 years.

⇒ The accumulated value of the first six payments after six years is

$$100 \cdot s_{\overline{6}|\,0.05}$$

At the end of the 10 years, having accrued interest at a rate of 4% during the last 4 years, this value grows to

$$100 \cdot s_{\overline{6}|\,0.05} \cdot (1.04)^4$$

Meanwhile, the accumulated value of the last 4 payments is

$$100 \cdot s_{\overline{4}|\,0.04}$$

So, the final answer is

$$100\left[s_{\overline{6}|\,0.05} \cdot (1.04)^4 + s_{\overline{4}|\,0.04}\right] = 1251.43$$

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