Level Annuities with Payments Less Frequent than Each Interest Period

1. Annuity-immediate
2. Annuity-due

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Symoblic approach

- In this chapter we have to distinguish between payment periods and interest periods
- Consider a basic annuity that lasts for n **interest periods**, and has r payments where $n = r \cdot k$ for some integer k
- In other words, this annuity has a payment at the end of each k interest periods
- i ... the **effective interest rate** per interest period
- I ... the **effective interest rate** per payment period, i.e.,

$$I = (1+i)^k - 1$$

- Then, the value at issuance of this annuity is $a_{\overline{r}|I}$ and

$$a_{\overline{r}|I} = \frac{1-(1+I)^{-r}}{I} = \frac{1-(1+i)^{-rk}}{(1+i)^k - 1} = \frac{a_{\overline{n}|i}}{s_{\overline{k}|i}}$$

- The accumulated value is $s_{\overline{r}|I} = \frac{s_{\overline{n}|i}}{s_{\overline{k}|i}}$

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An Example

- Find an expression in terms of symbols of the type $a_{\overline{n}|}$ and $s_{\overline{n}|}$, for the present value of an annuity in which there are a total of r payments of 1. The first payment is to be made 7 years from today, and the remaining payments happen at three year intervals.

$\Rightarrow$ The present value of this annuity can be expressed in terms of the annual discount factor as

$$v^7 + v^{10} + v^{13} + \cdots + v^{3r+4}$$

Calculating the partial sum of the geometric series above, we get

$$\frac{v^7 - v^{3r+7}}{1 - v^3} = \frac{-(1 - v^7) + (1 - v^{3r+7})}{1 - v^3} = \frac{-\frac{1-v^7}{i} + \frac{1-v^{3r+7}}{i}}{\frac{1-v^3}{i}} = \frac{a_{\overline{3r+7}|} - a_{\overline{7}|}}{a_{\overline{3}|}}$$

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An Example: Unknown final payment

- An investment of \$1000 is used to make payments of \$100 at the end of each year for as long as possible with a smaller final payment to be made at the time of the last regular payment. If interest is 7% convertible semiannually, find the number of payments and the amount of the total final payment.

An Example: Unknown final payment (cont'd)

⇒ Using the expression for the present value of this annuity, we get the equation of value at time 0

$$100 \cdot \frac{a_{\overline{n}|0.035}}{s_{\overline{2}|0.035}} = 1000$$

where n denotes the unknown number of regular interest periods that the annuity lasts.

The equation of value yields

$$a_{\overline{n}|0.035} = 10 \cdot s_{\overline{2}|0.035} = 20.35$$

We get that $n = 36$ and that 18 regular payments and an additional smaller payment must be made.

Let R denote the amount of the smaller final payment. Then, the time n equation of value reads as

$$R + 100 \cdot \frac{s_{\overline{36}|0.035}}{s_{\overline{2}|0.035}} = 1000 \cdot (1.035)^{36}$$

Thus, $R = \$10.09$

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Value at issuance and accumulated value

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- Then, the value at issuance of this annuity-due is $\ddot{a}_{\overline{r}|\,I}$ and

$$\ddot{a}_{\overline{r}|\,I} = (1+I) \cdot a_{\overline{r}|\,I} = \frac{a_{\overline{n}|\,i}}{a_{\overline{k}|\,i}}$$

- Similarly, we get that the accumulated value equals $\ddot{s}_{\overline{r}|\,I} = \frac{s_{\overline{n}|\,i}}{a_{\overline{k}|\,i}}$
- *Caveat:* The above accumulated value is k interest conversion periods after the last payment ...

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An Example: Accumulated value

- Find the accumulated value at the end of four years of an investment fund in which \$100 is deposited at the beginning of each quarter for the first two years and \$200 is deposited at the beginning of every quarter for the second two years. Assume that the fund earns 12% convertible monthly.

⇒ The rate of interest is 1% per month. In this annuity-due, there are 48 interest periods and each payment period consists of 3 interest coversion periods. So, the accumulated value is

$$100 \cdot \frac{s_{\overline{48}|\,0.01} + s_{\overline{24}|\,0.01}}{a_{\overline{3}|\,0.01}} = 100 \cdot \frac{61.2226 + 26.9735}{2.9410} = \$2999$$

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