On some special nonlevel annuities and yield rates for annuities

1. Annuities with payments in geometric progression
2. Annuities with payments in Arithmetic Progression

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An Example

Assume that an annuity-immediate provides 20 annual payments: the first payment being \$1,000. The payments increase in such a way that each payment is 4% greater than the preceding payment. Find the present value of this annuity at an annual effective rate of interest of 7%.

$\Rightarrow$ Let us look at a more general situation.

Let n denote the numer of periods.

For simplicity, assume that the first payment is equal to 1, and let all subsequent payments increase in geometric progression with the common ratio equal to $1+g$

Then, the present value of this annuity equals

$$\begin{aligned}
PV &= v + v^2 \cdot (1+g) + v^3 \cdot (1+g)^2 + \cdots + v^n \cdot (1+g)^{n-1} \\
&= v \cdot \left[1 + v \cdot (1+g) + (v \cdot (1+g))^2 + \cdots + (v \cdot (1+g))^{n-1}\right] \\
&= v \cdot \left[\frac{1-(v \cdot (1+g))^n}{1-v(1+g)}\right] \\
&= \frac{1-\left(\frac{1+g}{1+i}\right)^n}{i-g}
\end{aligned}$$

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In our case, the first payment is equal to \$1,000, so we have to multiply the result by that sum.

We get that the present value of the annuity from the problem equals

$$1000 \cdot \frac{1 - (\frac{1+0.04}{1+0.07})^{20}}{0.07 - 0.04} = 14,459$$

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A simplifying formula

- Let us revisit the formula for the present value:

$$PV = \frac{1 - \left(\frac{1+g}{1+i}\right)^n}{i - g}$$

- Define $j = (i - g)/(1 + g)$. Then, we can express the above present value as

$$(1 + i)^{-1} \cdot \ddot{a}_{\overline{n|}\,j}$$

- In particular, if $i = g$ (i.e., if the increase in the payment exactly offsets the interest rate), then the present value becomes

$$\frac{n}{1 + i}$$

- *Assignment:* Example 3.8.3 in the textbook;
  Problems 3.8.1,2,3,4,5 from the textbook
- Now, let us solve Sample FM Problems #11 and #16 together . . .

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The Set-up

- n ... the number of time periods for the annuity-immediate
- P ... the value of the first payment
- Q ... the amount by which the payment per period **increases**
- So, the payment at the end of the j^{th} periods is

$$P + Q(j-1)$$

- $(I_{P,Q}\,a)_{\overline{n|}\,i}$ ... the present value of the annuity described above
- $(I_{P,Q}\,s)_{\overline{n|}\,i}$ ... the accumulated value at the time of the last payment of the annuity described above

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Formulas for the accumulated and present values

Simple algebra yields:

$$(I_{P,Q}\, s)_{\overline{n}|\, i} = P \cdot s_{\overline{n}|\, i} + \frac{Q}{i} \cdot (s_{\overline{n}|\, i} - n)$$

$$(I_{P,Q}\, a)_{\overline{n}|\, i} = P \cdot a_{\overline{n}|\, i} + \frac{Q}{i} \cdot (a_{\overline{n}|\, i} - n \cdot v^n)$$

- In particular, if $P = Q = 1$, the notation and the equations can be simplified to

$$(Is)_{\overline{n}|\, i} = \frac{\ddot{s}_{\overline{n}|\, i} - n}{i}$$

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Formulas for the accumulated and present values
Decreasing payments

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- *Assignment:* Examples 3.9.8, 3.9.9
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An Example

- Find the expression for the present value of an annuity-immediate such that payments start at the amount of 1 dollar, increase by annual amounts of 1 to a payment of n, and then decrease by annual amounts of 1 to a final payment of 1.
  You are allowed to use the standard notation for present values of basic annuities-immediate.

$\Rightarrow$ The present value is

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\begin{aligned}
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An Example: A Perpetuity-Immediate

- Find the present value of a perpetuity-immediate whose successive payments are

$$1, 2, 3, \dots$$

at an effective per period interest rate of 0.05.

⇒ If we take a limit as $n \to \infty$ in the formula

$$(Ia)_{\overline{n}|\,i} = \frac{\ddot{a}_{\overline{n}|\,i} - n \cdot v^n}{i} = \frac{(1 + a_{\overline{n-1}|\,i}) - n \cdot v^n}{i},$$

we get

$$(Ia)_{\overline{\infty}|\,i} = \frac{1}{i} + \frac{1}{i^2} = \frac{1}{0.05} + \frac{1}{0.05^2} = 420$$

- Let us look at Sample FM Problems #18 and #6 ...

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