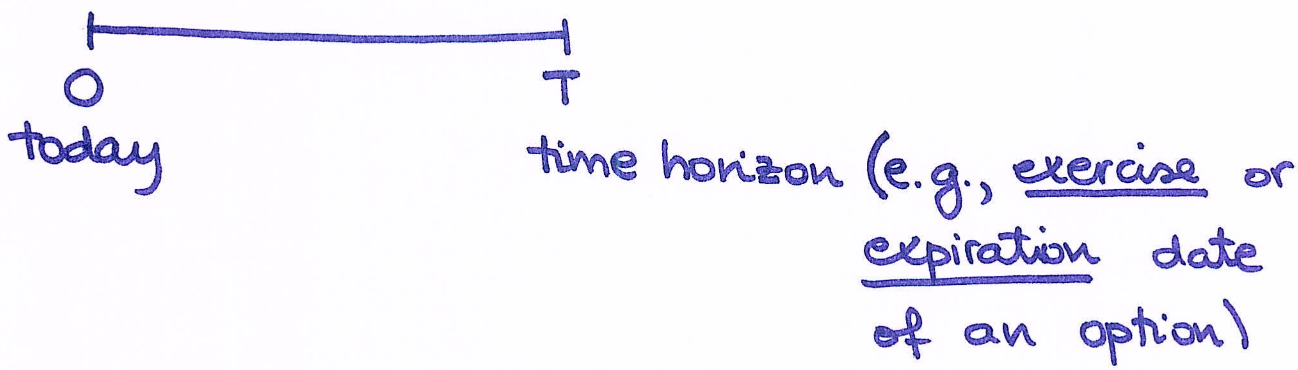
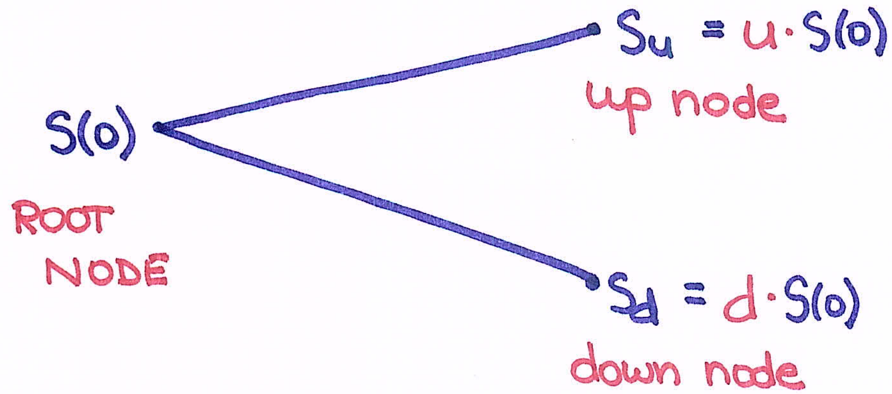
The Binomial Asset-Pricing Model.

$S(0)$ ... the initial stock price (observable!)





By convention:

$S_u > S_d$

$\Downarrow$

$u > d$

h ... length of a single period

$S(T) = S(h)$ ... rnd variable denoting the time-T stock price w/ two possible values: S_u & S_d

u & d completely describe our stock-price model

An interpretation:

$$u = \frac{S_u}{S(0)} = \frac{S_u - S(0)}{S(0)} + 1$$

$$d = \frac{S_d}{S(0)} = \frac{S_d - S(0)}{S(0)} + 1$$

Simple rate of return

Market Model:

- riskless asset : @ the continuously compounded risk-free interest rate r
- risky asset : continuous-dividend-paying stock w/ δ ... dividend yield

Consider investing in one share of stock @ time 0

WEALTH

$S(0)$ → $S_u = u \cdot S(0)$ $\qquad e^{\delta \cdot h} \cdot S_u = e^{\delta \cdot h} \cdot u \cdot S(0)$

$S(0)$ → $S(0)e^{rh}$ (risk-free investment)

$S(0)$ → $S_d = d \cdot S(0)$ $\qquad e^{\delta \cdot h} \cdot S_d = e^{\delta \cdot h} \cdot d \cdot S(0)$

$\underbrace{\qquad\qquad}_{h}$

⇒ We suspect that the no-arbitrage condition is: $e^{\delta \cdot h} \cdot d \cdot \cancel{S(0)} < e^{r \cdot h} \cdot \cancel{S(0)} < e^{\delta \cdot h} \cdot u \cdot \cancel{S(0)}$

$$\boxed{d < e^{(r-\delta)\cdot h} < u} \qquad (2.)$$

* Assume, to the contrary, that

$$e^{(r-\delta)\cdot h} \leq d < u$$

I. Suspicion. ✓

II. Propose an arbitrage portfolio:

- long one share of stock

III. Verification.

Initial Cost: $S(0)$

Payoff: $e^{\delta \cdot h} \cdot S(h)$ ... a rnd variable

$\Rightarrow$ Profit: $e^{\delta \cdot h} \cdot S(h) - S(0) e^{rh}$

The two possible states of the world:

"up" node: $e^{\delta \cdot h} \cdot u \cdot S(0) - e^{r \cdot h} \cdot S(0) > 0$

"down" node: $e^{\delta \cdot h} \cdot d \cdot S(0) - e^{r \cdot h} \cdot S(0) \geq 0$

$\Downarrow$

This is, indeed, an arbitrage portfolio!

Binomial Option Pricing.

· **Stock price tree** :

$$u > d$$

$$u > e^{(r-\delta)h} > d$$

$$\begin{tikzcd}[row sep=small] & S_u = u \cdot S(0) \\ S(0) \arrow[ur] \arrow[dr] & \\ & S_d = d \cdot S(0) \end{tikzcd}$$

$\xrightarrow[\text{the tree}]{\text{populating}}$

· We are interested in pricing European-style derivative securities w/ exercise date @ the end of the period. It is completely dermined by its PAYOFF FUNCTION : $v(\cdot)$

e.g., for a call $v(s) = (s-K)_+$
for a put $v(s) = (K-s)_+$

$\Rightarrow$ The payoff of this derivative security is a random variable given by

$$V(T) := v(S(T)) \; (= v(S(h)))$$

using the payoff f'tion — one period

$$\begin{tikzcd}[row sep=small] & S_u \\ \begin{array}{c} S(0) \\ \boxed{V(0) = ?} \end{array} \arrow[ur] \arrow[dr] & \\ & S_d \end{tikzcd}$$

PAYOFF

$V_u := v(S_u)$ e.g., for a call $V_u = (S_u - K)_+$

$V_d := v(S_d)$ $V_d = (S_d - K)_+$

$\longrightarrow$

$\xleftarrow{\text{PRICING!}}$

For all derivative securities, we can replicate them in this simple model by:

- Δ shares of stock
- B invested @ the risk-free rate (r)

STOCK · PRICE TREE

PAYOFF — REPLICATING PORTFOLIO'S WORTH

$$
\begin{array}{ccccc}
 & S_u & V_u & = & \Delta e^{\delta \cdot h} \cdot S_u + Be^{r \cdot h} \\
S(0) & & & & \\
 & S_d & V_d & = & \Delta e^{\delta \cdot h} \cdot S_d + Be^{r \cdot h} \\
\end{array}
\quad -
$$

$\underbrace{\qquad}_{h}$

$\Delta = ?$, $B = ?$

$$\Delta e^{\delta \cdot h}(S_u - S_d) = V_u - V_d$$

$$\boxed{\Delta = e^{-\delta \cdot h} \cdot \frac{V_u - V_d}{S_u - S_d}} \quad ✓$$

$$Be^{r \cdot h} = V_u - \frac{V_u - V_d}{\underbrace{S_u - S_d}_{\cancel{S(0)}(u-d)}} \cdot \overbrace{S_u}^{=\cancel{S(0)} \cdot u}$$

$$= \frac{\cancel{u \cdot V_u} - d \cdot V_u - \cancel{u \cdot V_u} + u \cdot V_d}{u-d}$$

$$\boxed{B = e^{-r \cdot h} \cdot \frac{u \cdot V_d - d \cdot V_u}{u-d}} \quad ✓$$

⇒ The option price @ time 0 is:

$$\boxed{V(0) = \Delta \cdot S(0) + B}$$

Pricing by Replication