Binomial Option Pricing [cont'd].

D: April 10^{th}, 2019.

The one-period stock-price tree:

$\overbrace{\qquad\qquad}^{h=T}$ **PAYOFF:** Replicating Portfolio:

$$S(0) \begin{cases} S_u = u \cdot S(0) & V_u := v(S_u) = \Delta e^{\delta \cdot h} \cdot S_u + Be^{rh} \\ S_d = d \cdot S(0) & V_d := v(S_d) = \Delta e^{\delta \cdot h} \cdot S_d + Be^{rh} \end{cases}$$

$\xrightarrow[\text{prices}]{\text{stock}}$

The no-arbitrage condition:

$$\boxed{d < e^{(r-\delta)h} < u}$$

In this model, we can replicate any derivative security using a portfolio w/ the following structure:

- Δ shares of stock
- B invested @ the risk-free rate r

We arrive @ the following system of eq'ns:

$$\begin{array}{r} \Delta e^{\delta \cdot h} \cdot S_u + Be^{rh} = V_u \\ - \quad \Delta e^{\delta \cdot h} \cdot S_d + Be^{rh} = V_d \\ \hline \end{array}$$

$$\boxed{\Delta = e^{-\delta \cdot h} \cdot \frac{V_u - V_d}{S_u - S_d}}$$

unitless ✓

$$\Rightarrow \quad \frac{V_u - V_d}{S_u - S_d} \cdot S_u + B e^{rh} = V_u$$

$$\Rightarrow \quad B e^{rh} = V_u - \frac{V_u - V_d}{(u-d)\cdot \cancel{S(0)}} \cdot u \cdot \cancel{S(0)}$$

$$\Rightarrow \quad \boxed{B = e^{-r \cdot h} \cdot \frac{u \cdot V_d - d \cdot V_u}{u-d}}$$

"in \$" ✓

By the law of the one price:

$$\boxed{V(0) = \Delta \cdot S(0) + B}$$

PRICING BY REPLICATION :)

$r = 0.04$

Problem 4.18. Let the continuously compounded risk-free interest rate be equal to 0.04. Consider a one-period binomial tree with every period of length one year used to model the stock price of a stock whose current price is \$80 per share. The stock pays dividends continuously with a dividend yield of 0.02. In the model, it is assumed that the stock price can either go up by \$5 or down by \$4.

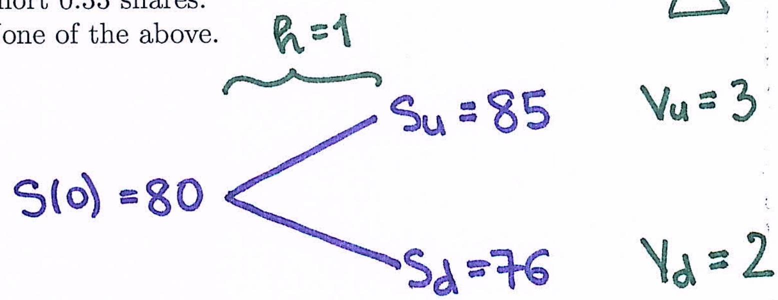
$\delta =$ 0.02

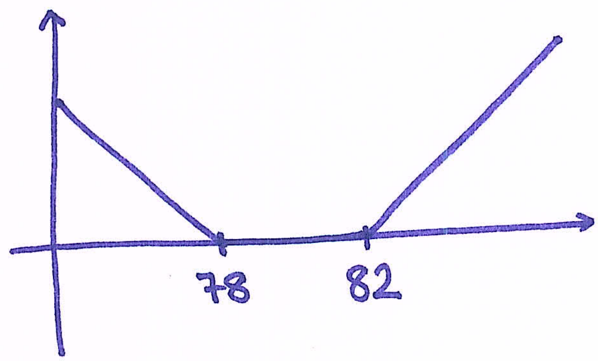
You use the binomial tree to construct a replicating portfolio for a (78, 82)−strangle on the above stock. What is the stock investment in the replicating portfolio?

$\Delta = ?$

(a) Long 0.1089 shares.
(b) Long 0.33 shares.
(c) Short 0.1089 shares.
(d) Short 0.33 shares.
(e) None of the above.

$$\Delta = e^{-\delta \cdot h} \cdot \frac{V_u - V_d}{S_u - S_d}$$





$$\Rightarrow \Delta = e^{-0.02} \cdot \frac{3-2}{85-76} = \ldots = 0.108911 \Rightarrow \text{(a)}$$ ∎

$r = 0.04$

Problem 4.19. Let the continuously compounded risk-free interest rate be equal to 0.04. Consider a one-period binomial tree with every period of length one year used to model the stock price of a non-dividend-paying stock whose current price is \$50 per share. In the model, it is assumed that the stock price can either go up by 5%, or down by 10%.

$h = 1$

$\delta = 0$

You use the binomial tree to construct a replicating portfolio for a (45, 55)−call bull spread on the above stock. What is the risk-free investment in the replicating portfolio?

(a) Borrow \$45
(b) Borrow \$43.24
(c) Lend \$45
(d) Lend \$43.24
(e) None of the above.

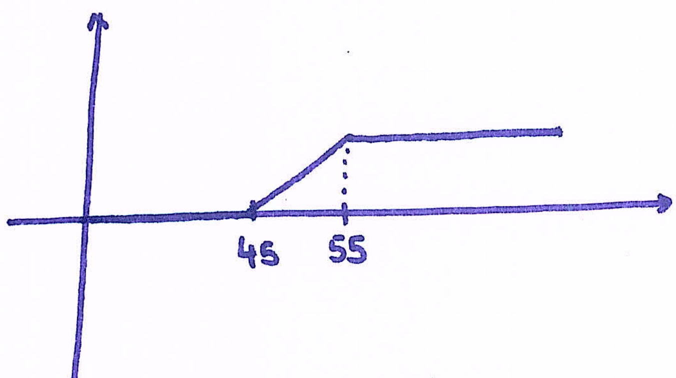
$B = ?$

$S_u = 52.5$ $\quad V_u = 7.5$

$S(0) = 50$

$S_d = 45$ $\quad V_d = 0$

(45, 55) · call bull spread



$$B = e^{-r\cdot h} \cdot \frac{u V_d - d \cdot V_u}{u - d}$$

$$B = e^{-0.04} \cdot \frac{1.05(0) - 0.9 \cdot (7.5)}{1.05 - 0.9} = -43.2355$$

$\Downarrow$ Borrow!

$\Downarrow$ (B)

4.

$$V(0) = \Delta \cdot S(0) + B$$

$$V(0) = e^{-\delta \cdot h} \cdot \frac{V_u - V_d}{\underbrace{S_u - S_d}_{= \cancel{S(0)}(u-d)}} \cdot \cancel{S(0)} + e^{-r \cdot h} \cdot \frac{u V_d - d V_u}{u-d}$$

$$V(0) = \frac{1}{u-d}\left[e^{-\delta \cdot h}(V_u - V_d) + e^{-rh} \cdot (uV_d - dV_u)\right]$$

$$= e^{-r \cdot h} \cdot \frac{1}{u-d}\left[e^{(r-\delta)h}(V_u - V_d) + uV_d - dV_u\right]$$

$$= e^{-r \cdot h} \cdot \frac{1}{u-d}\left[V_u\left(e^{(r-\delta)h} - d\right) + V_d\left(u - e^{(r-\delta)h}\right)\right]$$

$$V(0) = e^{-r \cdot h}\left[V_u \cdot \frac{e^{(r-\delta)h} - d}{u-d} + V_d \cdot \frac{u - e^{(r-\delta)h}}{u-d}\right]$$

ADD UP TO 1 !!!

BOTH POSITIVE !!!

Choose to interpret the two terms as probabilities, i.e., define

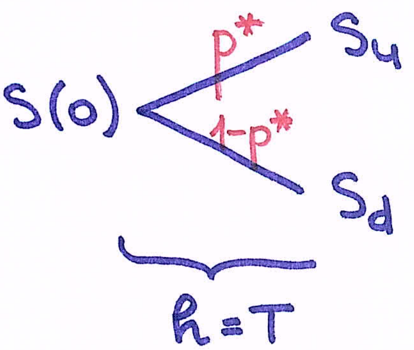
$$p^* := \frac{e^{(r-\delta)h} - d}{u-d}$$

... THE RISK·NEUTRAL PROBABILITY of the stock price going up in a single period.

$\Rightarrow$ THE RISK·NEUTRAL PRICING FORMULA

$$V(0) = e^{-r \cdot T}\left[p^* \cdot V_u + (1-p^*) \cdot V_d\right] = e^{-rT}\,\mathbb{E}^*[V(T)]$$

Example. Consider a stock paying dividends continuously w/ dividend yield δ



PAYOFF:

$V_u = S_u$

$V_d = S_d$

Q: How much must we pay today, according to our model, in order to receive one share of stock @ time T?

By risk-neutral pricing:

$$V(0) = e^{-r\cdot T}\cdot \mathbb{E}^*[V(T)] = e^{-r\cdot T}\left[p^*\cdot S_u + (1-p^*)\cdot S_d\right]$$

$$= e^{-r\cdot T}\left[\frac{e^{(r-\delta)h}-d}{u-d}\cdot \underbrace{S_u}_{=S(0)\cdot u} + \frac{u-e^{(r-\delta)h}}{u-d}\cdot \underbrace{S_d}_{=S(0)\cdot d}\right]$$

$$= e^{-r\cdot T}\cdot\frac{1}{u-d}\left[u\cdot \underbrace{e^{(r-\delta)h}} - \cancel{u\cdot d} + \cancel{u\cdot d} - d\cdot \underbrace{e^{(r-\delta)h}}\right]\cdot S(0)$$

$$\underset{\substack{\uparrow\\ h=T}}{=} e^{-r\cdot T}\cdot e^{(r-\delta)\cdot T}\cdot\frac{1}{\cancel{u-d}}\left[\cancel{u-d}\right]\cdot S(0)$$

$$= e^{-\delta\cdot T}\cdot S(0) \underset{\substack{\uparrow\\ \text{Consistency!}}}{=} F^P_{0,T}(S)$$

Example. Look @ the following binomial tree for a stock price S:

$$
\begin{tikzcd}[column sep=large]
 & S_u = u\cdot S(0) \qquad V_u = S_u - K \\
S(0) \arrow[ur, "p^*"] \arrow[dr, "1-p^*"'] & \\
 & S_d = d\cdot S(0) \qquad V_d = 0
\end{tikzcd}
$$

$$\underbrace{\qquad\qquad}_{h=T}$$

Consider a time-T, strike-K European call w/ $\boxed{S_d < K < S_u}$

(the only interesting case :)

$$
\Rightarrow \begin{cases} \Delta_C = e^{-\delta\cdot h}\cdot \dfrac{V_u}{S_u - S_d} = e^{-\delta\cdot h}\cdot \dfrac{S_u - K}{S_u - S_d} \in [0,1] \\[2ex] B_C = e^{-r\cdot h}\cdot \dfrac{\cancel{u\cdot \overset{=0}{V_d}} - d\cdot V_u}{u-d} = - e^{-r\cdot h}\cdot d \cdot \dfrac{V_u}{u-d} \end{cases}
$$

$\Downarrow$

Borrowing!

$$\boxed{V_C(0) = e^{-r\cdot T}\cdot p^* \cdot V_u}$$

■