Example. [Rebate option]
Point of $R \text{ at } t = T$, if the barrier $H$ was ever crossed/touched by the stock price during $[0, T]$.

1. If $S(0) < H$, then rebate if $M(T) \geq H$
   
   $w/ \ M(T) := \max_{0 \leq t \leq T} S(t)$
   
   $\Rightarrow \text{PAYOFF: } V(T) = R \cdot \mathbb{I}_{[M(T) \geq H]}$

2. If $S(0) > H$, then rebate if $m(T) \leq H$
   
   $w/ \ m(T) := \min_{0 \leq t \leq T} S(t)$
   
   $\Rightarrow \text{PAYOFF: } V(T) = R \cdot \mathbb{I}_{[m(T) \leq H]}$
Exercise. Consider a non-dividend-paying stock with \( S(0) = 100 \) and whose volatility is 0.3. We use a two-period forward binomial tree to model the stock price over the following year. Assume that the continuously compounded, risk-free interest rate equals 0.08.

A rebate option pays $20 @ time-1 if the stock price ever reaches the level $110 during the following year. What is the price of this rebate option?

A *forward* tree:

\[
\begin{align*}
\text{u} & = e^{(r - \sigma^2)\Delta t + \sigma \sqrt{\Delta t}} = e^{0.08 \cdot \frac{1}{2} + 0.3 \sqrt{\frac{1}{2}}} = 1.2868 \\
\text{d} & = e^{(r - \sigma^2)\Delta t - \sigma \sqrt{\Delta t}} = e^{0.04 - 0.3 \sqrt{\frac{1}{2}}} = 0.8419
\end{align*}
\]

\[
\begin{align*}
S(0) = 100 & \quad \text{p}^* & \quad \text{Su} = 128.68 \\
 & \quad \text{Sd} & \quad \text{Sd} \text{d} = 108.33
\end{align*}
\]

\[
p^* = \frac{1}{1 + e^{\sigma \sqrt{\Delta t}}} = \frac{1}{1 + e^{0.3 \sqrt{\frac{1}{2}}}} = 0.447
\]

Payoff:

\[
\begin{align*}
0 & \quad \text{w/ probab. } 1 - p^* \\
20 & \quad \text{w/ probab. } p^*
\end{align*}
\]

\[
V_R(0) = e^{-0.08 \cdot 20} \cdot 0.447 \approx 8.26
\]
Family of options:

- and -

up/down in/out call/put

\Rightarrow 8 \text{ specific types}

Example: An up-and-in call

w/ strike K and barrier H. Say, \( S(0) < H \).

In general: \( V(T) = (S(T) - K)_{+}I_{[M(T) \geq H]} \)

Example: An up-and-out put

PAYOFF: \( V(T) = (K - S(T))_{+}I_{[M(T) < H]} \)

Q: Consider an up-and-out call w/ \( K > H \).
   What can you say about it? PAYOFF=0
   \Rightarrow PRICE=0.

Q: Consider the following portfolio:

\{ up-and-in option \}

\{ up-and-out option \} OTHERWISE IDENTICAL
PAYOFF: With the payoff of the vanilla version of the option $V(T)$:

$$V(T) \cdot \mathbb{I}_{[M(T) \geq H]} + V(T) \cdot \mathbb{I}_{[M(T) < H]} =$$

$$= V(T) \left[ \mathbb{I}_{[M(T) \geq H]} + \mathbb{I}_{[M(T) < H]} \right] = V(T)$$

$\Rightarrow$ We had a replicating portfolio for the "regular" option.
2. You have observed the following monthly closing prices for stock XYZ:

<table>
<thead>
<tr>
<th>Date</th>
<th>Stock Price</th>
</tr>
</thead>
<tbody>
<tr>
<td>January 31, 2008</td>
<td>105</td>
</tr>
<tr>
<td>February 29, 2008</td>
<td>120</td>
</tr>
<tr>
<td>March 31, 2008</td>
<td>115</td>
</tr>
<tr>
<td>April 30, 2008</td>
<td>110</td>
</tr>
<tr>
<td>May 31, 2008</td>
<td>115</td>
</tr>
<tr>
<td>June 30, 2008</td>
<td>110</td>
</tr>
<tr>
<td>July 31, 2008</td>
<td>100</td>
</tr>
<tr>
<td>August 31, 2008</td>
<td>90</td>
</tr>
<tr>
<td>September 30, 2008</td>
<td>105</td>
</tr>
<tr>
<td>October 31, 2008</td>
<td>125</td>
</tr>
<tr>
<td>November 30, 2008</td>
<td>110</td>
</tr>
<tr>
<td>December 31, 2008</td>
<td>115</td>
</tr>
</tbody>
</table>

The following are one-year European options on stock XYZ. The options were issued on December 31, 2007.

(i) An arithmetic average Asian call option (the average is calculated based on monthly closing stock prices) with a strike of 100.

\[ A(T) = 100 + \frac{1}{12} \left( 5 + 20 + 15 + 10 + 15 + 10 + 0 \right) \]
\[ H_{A} = 125 \]
\[ K_{A} = 120 \]
\[ \text{PAYOFF : } (A(T) - K_{A})^{+} \]

(ii) An up-and-out call option with a barrier of 125 and a strike of 120.

\[ A(T) = 100 + \frac{5}{12} \left( 5 + 25 + 10 + 15 \right) \]
\[ H_{A} = 120 \]
\[ K_{A} = 120 \]
\[ \text{PAYOFF : 0} \]

(iii) An up-and-in call option with a barrier of 120 and a strike of 110.

\[ A(T) = 100 + \frac{120}{12} = 100 + 10 \]
\[ H_{A} = 120 \]
\[ K_{A} = 110 \]
\[ \text{Knocked in} \]
\[ \text{PAYOFF : 10} \]

Calculate the difference in payoffs between the option with the largest payoff and the option with the smallest payoff.

(A) 5
(B) 10
(C) 15
(D) 20
(E) 25
SAMPLE MFE

42. Prices for 6-month 60-strike European up-and-out call options on a stock \( S \) are available. Below is a table of option prices with respect to various \( H \), the level of the barrier. Here, \( S(0) = 50 \).

<table>
<thead>
<tr>
<th>( H )</th>
<th>Price of up-and-out call</th>
</tr>
</thead>
<tbody>
<tr>
<td>60</td>
<td>0</td>
</tr>
<tr>
<td>70</td>
<td>0.1294</td>
</tr>
<tr>
<td>80</td>
<td>0.7583</td>
</tr>
<tr>
<td>90</td>
<td>1.6616</td>
</tr>
<tr>
<td>( \infty )</td>
<td>4.0861</td>
</tr>
</tbody>
</table>

\[ \text{Q: Why is the price for } H = 60 \text{ exactly 0?} \]
\[ K = 60 \Rightarrow \text{PAYOFF} \]

\[ \text{Q: Why do the prices increase w/ respect to } H? \] It is less likely to get knocked out.

\[ \text{Q: What does it mean to set } H = \infty? \]

A "regular" call option.

Consider a special 6-month 60-strike European "knock-in, partial knock-out" call option that knocks in at \( H_1 = 70 \), and "partially" knocks out at \( H_2 = 80 \). The strike price of the option is 60. The following table summarizes the payoff at the exercise date:

\[
\begin{array}{c|c|c}
H_1 \text{ Not Hit} & H_1 \text{ Hit} \\
\hline
0 & 2 \times \max[S(0.5) - 60, 0] & \max[S(0.5) - 60, 0] \\
\end{array}
\]

Calculate the price of the option. Our only method is to construct a replicating portfolio consisting of the barrier options whose prices are given.

(A) 0.6289
(B) 1.3872
(C) 2.1455
(D) 4.5856
(E) It cannot be determined from the information given above.
$V_{SO}(T)$... the payoff of the special option

$V_{C}(T) = (S(T) - K)^+$

\[
V_{SO}(T) = \Pi_{[M(T) \geq H_1]} \left( V_{C}(T) + V_{C}(T) \Pi_{[M(T) < H_2]} \right)
\]

\[
V_{SO}(T) = (1 - \Pi_{[M(T) < H_2]}) \left( V_{C}(T) + V_{C}(T) \Pi_{[M(T) < H_2]} \right)
\]

Remaining steps:

- distribute (FOIL)
- identify and recognize the payoffs of the up-and-out calls
- calculate the price