Compound Options

Stock

Option

Compound Option

Derivative Security

- European exercise date $T_u$

- European exercise date $T_c$

$T_c \leq T_u$

The payoff of the compound option is paid:

$v(U(T_c))$

The value of the underlying option @ time-$T_c$. $v$...the payoff function of the compound option.

Focus on the following family of options:

\underline{ON} call/put \underline{ON} call/put
**The Table of Payoffs**

\[ K_c \]...strike of the **compound** option

<table>
<thead>
<tr>
<th>Underlying</th>
<th>Call</th>
<th>Put</th>
</tr>
</thead>
<tbody>
<tr>
<td>Compound</td>
<td>((V_{\text{Call}}(T_c) - K_c)_+)</td>
<td>((V_{\text{Put}}(T_c) - K_c)_+)</td>
</tr>
<tr>
<td>Call</td>
<td>((K_c - V_{\text{Call}}(T_c))_+)</td>
<td>((K_c - V_{\text{Put}}(T_c))_+)</td>
</tr>
</tbody>
</table>

**Put-Call Parity for compound options:**

\[
\text{Call on Call (0)} - \text{Put on Call (0)} = \underbrace{\text{Call (0)}} - K_c e^{-rT_c}
\]

\[
\text{Call on Put (0)} - \text{Put on Put (0)} = \underbrace{\text{Put (0)}} - K_c e^{-rT_c}
\]
Consider a non-dividend-paying stock with the initial price of $100. Assume that the continuously compounded risk-free interest rate equals 0.05.

There is an at-the-money European put option on the above stock with exercise date in 2 years. This option is currently traded at $11.54.

A compound call on the above put option is issued. Its exercise date is one year from today and its strike is 6. The price of this compound call is $7.18.

What is the value of a compound put option on the above vanilla option?

\[
\text{Put-call parity for compound options:} \quad \text{Call on Put}(o) - \text{Put on Put}(o) = \frac{\text{Put}(o) - K e^{-rT_c}}{1 + e^{-rT_c}}
\]

\[
\begin{align*}
7.18 & - ? = \frac{11.54 - 6 e^{-0.05 \cdot 1}}{6 e^{-0.05 \cdot 1}}
\end{align*}
\]

? = 1.35
The simplest binomial tree suitable for pricing compound options is a two-period tree:

Let's price a put on a strike $K_c$.

\[
V_u^u = e^{-r(T_u - T_c)} \left[ p^* \cdot V_{uu} + (1 - p^*) \cdot V_{ud} \right]
\]
\[
V_d^u = e^{-r(T_d - T_c)} \left[ p^* \cdot V_{ud} + (1 - p^*) \cdot V_{dd} \right]
\]

The possible payoffs of the compound option:

\[
V_u = (K_c - V_u^u)_+
\]
\[
V_d = (K_c - V_d^u)_+
\]

\[
\Rightarrow \quad V(0) = e^{-rT_c} \left[ p^* \cdot V_u + (1 - p^*) \cdot V_d \right]
\]

The above approach can be generalized to:

- $T_k$: the end of the $k$th period for some $k \leq n$,
- $T_n$: the end of the $n$th (final) period in the tree.
Currency Option pricing

- Underlying asset: FOREIGN CURRENCY [FC] (an exchange rate) w/ \( r_F \) ... continuously compounded, risk-free interest rate for the foreign currency
- Domestic currency \( r_D \) ... continuously compounded, risk-free interest rate for the domestic currency

Analogy: Foreign Currency \( \leftrightarrow \) Continuous-dividend-paying assets

\[ r_F \leftrightarrow \delta \]

Q: Does the same analogy work for binomial option pricing?

One-period \( x \) ... exchange rate

\[ x(0) \]

\[ u \cdot x(0) \]

\[ d \cdot x(0) \]

\[ h = T \]

\[ \Delta \ldots \text{the number of units of FC bought at time-0} \]

\[ B \ldots \text{the risk-free investment in the DC} \]

\[ p^* = \frac{e^{(r_F - r_D) \cdot h} - d}{u - d} \]

In general
Forward binomial tree:

\[
\begin{align*}
u &= e^{(r_d - r_f)h + \sigma \sqrt{h}} \\
d &= e^{(r_d - r_f)h - \sigma \sqrt{h}}
\end{align*}
\]

\[
p^* = \frac{1}{1 + e^{\sigma \sqrt{h}}}
\]

Still: it only depends on the volatility.
5. Consider a 9-month dollar-denominated American put option on British pounds. You are given that:

- \( T = \frac{3}{4} \)
- \( F = FC \) (underlying)
- \( x(0) = 1.43 \)
- \( K = 1.56 \)
- \( \sigma = 0.3 \)
- The US dollar continuously compounded risk-free interest rate is 8%.
- The British pound continuously compounded risk-free interest rate is 9%.

Using a three-period binomial model, calculate the price of the put.

\[ n = 3 \Rightarrow h = \frac{1}{4} \]

Q: Use the **FORWARD BINOMIAL TREE** (as the "default" tree)

\[ U = e^{(r_D - r_F) \cdot \frac{1}{4} + 0.3 \cdot \frac{1}{4}} = 1.16 \]
\[ d = e^{-0.025 \cdot 0.15} = 0.86 \]

The risk-neutral probability:

\[ p^* = \frac{1}{1 + e^{-0.09 \cdot \frac{1}{4}}} = 0.46 \]

\[ x(0) = 1.43 \]
\[ x_u = 1.66 > K \]
\[ x_d = 1.23 \]

Complete @ home.
Which tree? The default is\textbf{ forward} $K = 1.56$

\begin{align*}
u &= e^{(r_b - r_F)h} + \sigma \sqrt{h} = e^{(0.08 - 0.09) \cdot \frac{1}{4}} + 0.3 \cdot \sqrt{\frac{1}{4}} = 1.1589 \\
d &= e^{(r_b - r_F)h} - \sigma \sqrt{h} = e^{-0.01 \cdot \frac{1}{4}} - 0.15 = 0.8584
\end{align*}

Finish this problem @ home!
It is Sample MFE Problem #5 😊