Law of the Unique Price

Let A and B be two static portfolios. Assume $V_A(T) = V_B(T)$. Then $V_A(0) = V_B(0)$.

No arbitrage!

Def'n. Consider a European-style derivative security. A static portfolio w/ the same payoff as that of the derivative security is called its replicating portfolio.

Example. "long cash call + long cash put
" zero-coupon bond w/ $1 redemption"

Example. Build a replicating portfolio for a European call w/ strike K and exercise date T consisting exclusively of digital options.

→: The payoff of the European call:

$$V_c(T) = (S(T) - K)_+ = (S(T) - K)I_{[S(T) \geq K]} = S(T)I_{[S(T) \geq K]} - KI_{[S(T) < K]}$$
The payoff of an asset call:

\[ V_{AC}(T) = S(T) \mathbb{I}_{[ST > K]} \]

\[ \text{jump!} \]

\[ \Downarrow \text{ long 1 asset call} \quad \text{ and } \quad \text{short K cash calls} \]

\[ \Rightarrow \quad V_c(0) = V_{AC}(0) - K \cdot V_{cc}(0) \]
Repaid forward

Initial cost: \( F^p \)
Payoff: \( S(T) \)

Forward

Initial cost: 0
Payoff: \( S(T) - F \)

Proposing to replicate a prepaid forward with:
- one long forward contract
- invest \( PV_{0,T}(F) \) @ the risk-free rate

This is a static portfolio!
Its payoff is: \( S(T) - F + PV_{0,T}(PV_{0,T}(F)) \)

\[ \Rightarrow \text{This is indeed a replicating portfolio for a prepaid forward!} \]

\[ \Rightarrow F^p = PV_{0,T}(F) \]
\[ \iff F = FV(F^p) \]
Focus on forwards on stocks:

**Case #1. Non-dividend-paying**

- **Outright Purchase**
- **Prepaid Forward**

Init. Cost: $S(0)$

I.C.: $F^p$

Payoff: $S(T)$

\[
F^p_{0,T}(S) = S(0)
\]

\[
F_{0,T}(S) = FV_{0,T}(S(0)) = S(0)e^{rT}
\]

**Case #2. Continuous dividends.** *(S... the dividend yield)*

*Continuous reinvestment of dividends*

- **Outright Purchase**
- **Prepaid Forward**

I.C.: $e^{-rT}S(0)$

I.C.: $F^p$
\[ F_{0,T}^p (S) = S(0) e^{-\delta T} \]
\[ F_{0,T} (S) = S(0) e^{-\delta T} e^{rT} \]

\[ F_{0,T} (S) = S(0) e^{(r-\delta)T} \]

Q: Assume \( r > \delta > 0 \).

Nominal amounts of cashflows are ordered as:

Prepaid forward < Outright Purchase < Forward < Fully Leveraged