5.1. **Butterfly spreads and convexity.** Please, provide your **complete solution** to the following problem(s):

**Problem 5.1.** (5 points) In a certain market, you are given that
the price of a 40−strike European call option on an underlying asset $S$ and with maturity $T$ is $11$;
the price of a 50−strike European call option on an underlying asset $S$ and with maturity $T$ is $6$;
the price of a 55−strike European call option on an underlying asset $S$ and with maturity $T$ is $3$.

Let the risk-free interest rate be $r = 0.05$.

A trader decides to construct the following portfolio:
(1) long one 40−strike call option;
(2) short three 50−strike European call options;
(3) long two 55−strike calls.

Suppose that at time $T = 1$ the value of the asset $S$ is $S(1) = 52$. What is the profit of the portfolio at time $T$?

**Solution:** The initial cost is
$$11 - 3 \cdot 6 + 2 \cdot 3 = -1.$$  

*Note:* The negative cost of $-1$ means that the trader can invest the one dollar surplus from the above trade in the money-market, i.e., lend the one dollar.

The payoff at maturity is
$$(S(T) - 40) - 3(S(T) - 50) + 2 \cdot 0 + 1 \cdot e^{rT} = e^{rT} + 2(55 - S(T)) \approx 7.05.$$  

Provide your **final answer** only for the following problems.

**Problem 5.2.** (5 points) Let $K_1 = 50, K_2 = 55$ and $K_3 = 65$ be the strikes of three European call options on the same underlying asset and with the same expiration date. Let $V_C(K_i)$ denote the price at time−0 of the option with strike $K_i$ for $i = 1, 2, 3$.

We are given that $V_C(K_1) = 16$ and $V_C(K_3) = 1$. What is the maximum possible value of $V_C(K_2)$ which still does not violate the convexity property of option prices?

(a) $V_C(K_2) = 10$
(b) $V_C(K_2) = 11$
(c) $V_C(K_2) = 13$
(d) $V_C(K_2) = 15$
(e) None of the above.

**Solution:** (b)

Let $V_C(K_2) = C$. The convexity requirement for call-option prices is
$$\frac{16 - C}{55 - 50} \geq \frac{C - 1}{65 - 55}.$$  

So,
$$10(16 - C) \geq 5(C - 1) \implies 33 \geq 3C \implies 11 \geq C.$$  

**INSTRUCTOR:** Milica Ćudina
**Problem 5.3.** (5 points) You are interested in purchasing a European call option on the underlying asset $S$ which has expiration date $T = 1$ year and strike $K = 70$. Denote the price of this call by $C$.

You are given that the premiums for European call options with the same expiration date and the same underlying asset but with the strikes $K_1 = 60$ and $K_2 = 75$ are $C_1 = 10$ and $C_2 = 3$, respectively.

You know that there is no arbitrage in your market. Find the maximal price $C$ that you might be charged so that none of the convexity inequalities are violated.

(a) $C = 16/3$
(b) $C = 6$
(c) $C = 25/3$
(d) $C = 28/3$
(e) None of the above.

Solution: (a)

\[ C = 10 \times \frac{1}{3} + 3 \times \frac{2}{3} = \frac{16}{3} \]

**Problem 5.4.** (5 points) Source: Problem 9.11 from “Derivatives Markets” by McDonald.

Suppose that the observed European call and European put prices are given in the following table:

<table>
<thead>
<tr>
<th>Strike</th>
<th>80</th>
<th>100</th>
<th>105</th>
</tr>
</thead>
<tbody>
<tr>
<td>Call premium</td>
<td>22</td>
<td>9</td>
<td>5</td>
</tr>
<tr>
<td>Put premium</td>
<td>4</td>
<td>21</td>
<td>24.80</td>
</tr>
</tbody>
</table>

Looking at the observed prices, we can suspect that there is an arbitrage opportunity in this market. Which of the following is indeed an arbitrage portfolio?

(a) Long one 80–strike call, short two 100–strike calls, long one 105–strike call
(b) Long one 80–strike call, short five 100–strike calls, long four 105–strike calls
(c) Short one 80–strike call, long five 100–strike calls, short four 105–strike calls
(d) Long one 80–strike call, short four 100–strike calls, long three 105–strike calls
(e) There is no arbitrage opportunity as a consequence of the above prices.

Solution: (b)

Note: All of the answers above are qualitative. In an exam you reason as we did in similar problems in class:
First, you realize that convexity inequalities are violated. This means that one should use the butterfly spreads to exploit the arbitrage opportunity. This observation immediately rules out (e) among the offered choices.

Second, you want to create a long butterfly spread to exploit arbitrage opportunities due to convexity violations. This observation rules out (c).

Third, we realize that the middle strike of 100 is 4/5 of the way from the smallest strike of 80 to the highest strike of 105. So, the symmetric butterfly spread in the offered choice (a) is ruled out as well.

Finally, again from he fact that $\lambda = 1/5$, we see that (b) is correct.

**Problem 5.5.** (5 points) Calculate the price of a long butterfly spread constructed using the following call options:

(1) a £3,925–strike call on the FTSE100 index which is being sold for £713.07;
(2) a £4,325–strike call on the FTSE100 index which is being sold for £496.46;
(3) a £4,725–strike call on the FTSE100 index which is being sold for £333.96.

Assume that the total number of the call options in your portfolio equals 4.

(a) £54.11
(b) £550.57
(c) £554.11
(d) £559.57
(e) None of the above

Instructor: Milica Čudina
Solution: (a)
The long butterfly spread can be constructed by buying a certain number of calls with the lowest strike, buying a certain number of calls with the highest strike and writing a certain number of calls with the middle strike. Note that in the present problem, the middle strike is exactly the average of the lowest and highest strikes, so we are dealing with a symmetric butterfly spread. This position is constructed by buying one each of the call with the “outer” strikes and writing two calls with the middle strike. The total cost equals
\[
713.07 + 333.96 - 2 \cdot 496.46 = 54.11.
\]

5.2. Collars. Please, provide your final answer only for the following problem(s):

**Problem 5.6.** (5 points) Please, solve the Sample FM(DM) Problem #59.

**Solution:** (e)

**Problem 5.7.** (5 points) Please, solve the Sample FM(DM) Problem #60.

**Solution:** (e)

**Problem 5.8.** (5 points) The future value in one year of the total costs of manufacturing a widget is $200. You will sell a widget in one year at its market price of $S(1).

Assume that the annual effective interest rate equals 5%, and that the current price of the widget equals $230.

You now purchase a one-year, $220-strike put on one widget for a premium of $7. You sell some of the potential gain by writing a one-year, $250-strike call on one widget for a $2 premium.

What is the range of the profit of your hedged portfolio?

(a) [14.75, 44.75]

(b) [220.75, 250.75]

(c) [220, 244.75]

(d) [220, 250]

(e) None of the above.

**Solution:** (a)
The payoff, written as a piecewise function, is
\[
v(s) = \begin{cases} 
K_P, & \text{for } 0 \leq s \leq K_P \\
s, & \text{for } K_P \leq s \leq K_C \\
K_C, & \text{for } K_C \leq s 
\end{cases}
\]

where \(K_P\) denotes the strike price for the put while \(K_C\) denotes the call’s strike price. So, the range of the payoff function is [220, 250].

The future value of the total cost of both production and hedging is
\[
200 + (7 - 2)(1 + 0.05) = 205.25.
\]

So, the range of the profit equals [14.75, 44.75].
5.3. **Ratio spreads.** Please, provide your **final answer only** for the following problem(s):

**Problem 5.9.** (5 points)
A strategy consists of longing a put on the market index with a strike of $830 and shorting a call option on the market index with a strike price of $830. The put premium is $18.00 and the call premium is $44.00. Interest rates are 0.5% effective per month. Determine the net profit if the index price at expiration is $830 (in 6 months).

(a) $0  
(b) $23.67 loss  
(c) $26.79 gain  
(d) $28.50 gain  
(e) None of the above.

**Solution:** (c)
In our usual notation, the expression for the payoff is

\[
(K - S(T))_+ - (S(T) - K)_+ = K - S(T).
\]

with \( K = 830 \). So, the payoff is 0 if \( S(T) = 830 \).
Since the profit calculation entails subtracting the future value of the initial cost, we have that the profit is

\[
0 - (1 + 0.005)^6(18 - 44) \approx 26.79.
\]

**Problem 5.10.** (5 points) Solve FM(DM) Problem #59.