
4.1. **Exchange options.** Provide your complete solution to the following problem:

**Problem 4.1.** (5 points) The minimum option

Let \( S = \{S(t), t \geq 0\} \) and \( Q = \{Q(t), t \geq 0\} \) denote the prices of two risky assets. The payoff of the minimum option is given by

\[
V_{\text{min}}(T) = \min(S(T), Q(T)).
\]

Propose a replicating portfolio consisting of prepaid forward contracts on \( S \) and/or \( Q \), and exchange options on \( S \) and \( Q \).

**Solution:**

\[
V_{\text{min}}(T) = \min(S(T), Q(T)) = S(T) + \min(0, Q(T) - S(T))
\]

\[= S(T) - \max(S(T) - Q(T), 0).\]

So, an example of a replicating portfolio is

- a prepaid forward contract on \( S \), and
- a short exchange call with \( S \) as underlying and \( Q \) as the strike asset

Other examples are

- a prepaid forward contract on \( S \), and
- a short exchange put with \( Q \) as underlying and \( S \) as the strike asset

- a prepaid forward contract on \( Q \), and
- a short exchange call with \( Q \) as underlying and \( S \) as the strike asset

- a prepaid forward contract on \( Q \), and
- a short exchange put with \( S \) as underlying and \( Q \) as the strike asset

**Problem 4.2.** (5 points) Let our market model include two continuous-dividend-paying stocks whose time-\( t \) prices are denoted by \( S(t) \) and \( Q(t) \) for \( t \geq 0 \). The current stock prices are \( S(0) = 160 \) and \( Q(0) = 80 \). The dividend yield for the stock \( S \) is \( \delta_S = 0.06 \) and the dividend yield for the stock \( Q \) is \( \delta_Q = 0.03 \).

The price of an exchange option giving its bearer the right to forfeit one share of \( Q \) for one share of \( S \) in one year is given to be $11.

Find the price of a maximum option on the above two assets with exercise date in a year. Remember that the payoff of the maximum option is \( \max(S(1), Q(1)) \).

**Solution:** As we showed in class

\[
V_{\text{max}}(0) = F_{0,1}^D(Q) + V_{\text{EC}}(0, S, Q) = Q(0)e^{-\delta_Q} + V_{\text{EC}}(0, S, Q) = 80e^{-0.03} + 11 = 88.64.
\]

Provide your final answer only to the following problems:

**Problem 4.3.** (2 points)

Exchange options are options where the underlying asset is an exchange rate. True or false?

**Solution:** FALSE
Problem 4.4. (2 points) Consider two European exchange options both with exercise date $T$, one that allows you to exchange a share of asset $S$ for a share of asset $Q$, and another one that allows you to forfeit a share of asset $Q$ and obtain a share of asset $S$ in return. Assume that neither asset pays any dividends.

As usual $V_{EC}(0, X(0), Y(0))$ stands for the time−0 price of the exchange call with the underlying asset $X$ and the strike asset $Y$.

Then, in our usual notation,

$$V_{EC}(0, S, Q) - V_{EC}(0, Q, S) = S(0) - Q(0).$$

Solution: TRUE

Generalized put-call parity.

Problem 4.5. (5 points) Assume that the continuously compounded interest rate equals 0.10.

Stock $S$ has the current price of $S(0) = 70$ and does not pay dividends. Stock $Q$ has the current price of $Q(0) = 65$ and it pays continuous dividends at the rate of 0.04.

An exchange option gives its holder the right to give up one share of stock $Q$ for a share of stock $S$ in exactly one year. The price of this option is $11.50.

Another exchange option gives its holder the right to give up one share of stock $S$ for a share of stock $Q$ in exactly one year. Find the price of this option.

(a) About $3.95
(b) About $11.10
(c) About $12.00
(d) About $14.25
(e) None of the above.

Solution: (a)

By the generalized put-call parity, we get the price we are looking for should be

$$V_{EC}(Q(0), S(0), 0) = V_{EC}(S(0), Q(0), 0) + F^P_{0,T}(Q) - F^P_{0,T}(S) = 11.50 + 65e^{-0.04} - 70 = 3.95.$$

Problem 4.6. (2 points) Consider two exchange options, one that allows you to exchange a share of asset $S$ for a share of asset $Q$, and another one that allows you to forfeit a share of asset $Q$ and obtain a share of asset $S$ in return.

Then, the prepaid forward prices of the two assets are the same if and only if the two exchange options have the same price.

Solution: TRUE

Problem 4.7. (2 points) Consider two exchange options, one that allows you to exchange a share of asset $S$ for a share of asset $Q$, and another one that allows you to forfeit a share of asset $Q$ and obtain a share of asset $S$ in return. Assume neither of the two stocks pays dividends.

Then, the current spot prices of the two assets are the same if and only if the two exchange options have the same price.

Solution: TRUE

Problem 4.8. (2 pts) Consider two European exchange options both with exercise date $T$, one that allows you to exchange a share of asset $S$ for a share of asset $Q$, and another one that allows you to forfeit a share of asset $Q$ and obtain a share of asset $S$ in return.

On the other hand, consider the maximum option with the payoff

$$V_{max}(T) = \max(S(T), Q(T)),$$

and the minimum option with the payoff

$$V_{min}(T) = \min(S(T), Q(T)).$$
Then, in our usual notation,
\[ V_{EC}(0, S, Q) + V_{EC}(0, Q, S) = V_{max}(0) + V_{min}(0). \]

**Solution:**  
**FALSE**  
If \( S(T) \leq Q(T) \), the payoff of a long exchange option allowing you to give up a unit of \( Q \) and receive a unit of \( S \) is
\[ V_{EC}(S(T), Q(T), T) = (S(T) - Q(T))_+ = 0, \]
i.e., the option goes unexercised. On the other hand, the payoff of a long exchange option allowing you to give up a unit of \( S \) and receive a unit of \( Q \) is
\[ V_{EC}(Q(T), S(T), T) = (Q(T) - S(T))_+ = Q(T) - S(T). \]
So, the payoff of the portfolio whose price is on the left-hand side of (4.2) is simply \( Q(T) - S(T) \).

The payoff of the portfolio whose initial cost is on the right-hand side of (4.2) is always \( S(T) + Q(T) \).

So, it is impossible for the proposed equality in prices to always be true.

**4.2. Monotonicity of option prices.**

**Problem 4.9.** (2 pts) Suppose that the European options with the same maturity and the same underlying assets have the following prices:

1. a 50−strike call costs $9;
2. a 55−strike call costs $10;

Then, some of the monotonicity conditions for no-arbitrage are violated by the above premiums.

**Solution:**  
**TRUE**  
We know that for strikes \( K_1 < K_2 \), the price of a call with strike \( K_1 \) should be greater than or equal to the price of a call with strike \( K_2 \) (see equation (9.13) in the book). This condition is violated.

**Problem 4.10.** (5 points)  
The current price of a share of stock \( S \) is $100. The stock is assumed to be paying a continuous dividend with the dividend yield of 0.04. Assume that the continuously compounded interest rate equals 0.05. Consider the following European gap options with the same exercise date in one year and the same underlying asset \( S \).

1. Gap call with strike price 100 and trigger price 100
2. Gap put with strike price 100 and trigger price 100
3. Gap call with strike price 100 and trigger price 110
4. Gap call with strike price 110 and trigger price 100
5. Gap call with strike price 100 and trigger price 80.

Which one of the above options has the highest price?

**Solution:**  
Let us try compare the prices of options I and II, first. Since for the both of them the trigger and the strike prices are the same, we are in fact dealing with just plain vanilla options. The “regular” put-call parity applies, and in our usual notation, we have
\[ V_I(0) - V_{II}(0) = F_{0,T}(S) - 100e^{-rT} = 100e^{-0.04} - 100e^{-0.05} = 100(e^{-0.04} - e^{-0.05}) > 0 \]
Option III has a lower price than option I since the payoff curve for option I dominates the payoff of option III.

Using the same type of comparison, we see that the value of option I is greater than the value of option IV (again, the payoff curve for option I is always above or at the same level as the payoff curve for option IV.
Option II has the higher price than option V (again, its payoff curve is always above or at the same level as the payoff curve for option V). So, the price for option II is higher than the price of option V.

We conclude that the option with the highest price of the ones offered is option I.

4.3. **Bear spreads.** Provide your final answer only to the following problem(s):

**Problem 4.11.** (2 points) A bear spread is a long position with respect to the underlying asset.

**Solution:** FALSE

**Problem 4.12.** (5 points) The following three nine-month European put options are available in the market:
- a $120-strike put with the premium of $12,
- a $127-strike put with the premium of $10,
- a $130-strike put with the premium of $14.

Which of the following positions certainly exploits the arbitrage opportunity caused by the above put premia?
- (a) Put bull spread.
- (b) Put bear spread.
- (c) Both of the above positions.
- (d) There is no arbitrage opportunity.
- (e) None of the above.

**Solution:** (c)

Please provide your **complete solutions** to the following problems.

**Problem 4.13.** (5 points) **Bear spreads in hedging**
An investor purchases a call option with an exercise price of $55 for $2.60. The same investor sells a call on the same asset with an exercise price of $60 for $1.40. At expiration, 3 months later, the asset price is $56.75. All other things being equal and given a continuously compounded annual interest rate of 4.0%, what is the profit to the investor?

**Solution:** The total initial cost of establishing the investor’s position is $2.60 − $1.40 = $1.20. The future value of this amount at expiration is $1.20e^{0.04\times0.25} = 1.20e^{0.01} \approx 1.21$.

The payoff at expiration is

\[(S(T) - 55)_+ - (S(T) - 60)_+ = 1.75.\]

So, the profit is $1.75 - 1.21 = 0.54$.

**Problem 4.14.** (6 points) **Source: “Derivatives markets” by McDonald**
Suppose that the European options with the same maturity and the same underlying assets have the following prices:
- (1) a 50-strike call costs $9;
- (2) a 50-strike put costs $7;
- (3) a 55-strike call costs $10;
- (4) a 55-strike put costs $6.

(i) (1 pts) Some of the monotonicity conditions for no-arbitrage are violated by the above premiums. Which ones?

(ii) (2 pts) Which spread positions would you use the create an arbitrage portfolio?

(iii) (3 pts) Substantiate your answer to part (ii) by demonstrating that using that particular spread one obtains a non-negative profit in all states of the world and strictly positive profit with positive probability.

**Solution:**
(i) We know that for strikes $K_1 < K_2$, the price of a call with strike $K_1$ should be greater than or equal to the price of a call with strike $K_2$ (see equation (9.13) in the book). This condition is violated. Moreover, the analogous relationship for prices of put options (see (9.14) in the textbook) is violated as well.

(ii) One can use a call bull spread (buy a 50–strike call and sell a 55–strike call) to profit from the violation of monotonicity of call prices. One can use a put bear spread (buy a 55–strike put and sell a 50–strike put) to profit from the violation of monotonicity of put prices.

(iii) In the case of the call bull spread, the cash flow can be summed up as follows:

<table>
<thead>
<tr>
<th>Expiration or Exercise</th>
<th>Transaction</th>
<th>$t = 0$</th>
<th>$S(T) &lt; 50$</th>
<th>$50 \leq S(T) \leq 55$</th>
<th>$S(T) &gt; 55$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Buy 50 strike call</td>
<td>$-9$</td>
<td>$0$</td>
<td>$S(T) - 50$</td>
<td>$0$</td>
<td>$55 - S(T)$</td>
</tr>
<tr>
<td>Sell 55 strike call</td>
<td>$+10$</td>
<td>$0$</td>
<td>$0$</td>
<td>$55 - S(T)$</td>
<td>$5 &gt; 0$</td>
</tr>
<tr>
<td>TOTAL</td>
<td>$+10$</td>
<td>$0$</td>
<td>$0$</td>
<td>$55 - S(T)$</td>
<td>$5 &gt; 0$</td>
</tr>
</tbody>
</table>

In the case of the put bear spread, the cash flow can be summed up as follows:

<table>
<thead>
<tr>
<th>Expiration or Exercise</th>
<th>Transaction</th>
<th>$t = 0$</th>
<th>$S(T) &lt; 50$</th>
<th>$50 \leq S(T) \leq 55$</th>
<th>$S(T) &gt; 55$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Buy 55 strike put</td>
<td>$-6$</td>
<td>$55 - S(T)$</td>
<td>$55 - S(T)$</td>
<td>$0$</td>
<td>$0$</td>
</tr>
<tr>
<td>Sell 50 strike put</td>
<td>$7$</td>
<td>$S(T) - 50$</td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
</tr>
<tr>
<td>TOTAL</td>
<td>$+1$</td>
<td>$5 &gt; 0$</td>
<td>$55 - S(T) &gt; 0$</td>
<td>$0$</td>
<td>$0$</td>
</tr>
</tbody>
</table>

Please note that we initially receive money, and that at expiration the profit is non-negative. We have, indeed, found arbitrage opportunities.