Prerequisite material. Transaction costs. Continuously compounded interest.

1.1. **Prerequisite material.** Please, provide your **final answer only** to the following problem.

**Problem 1.1.** (5 pts)

You invest an amount $A$ into an account at time $-0$. The account is governed by a continuously compounded risk-free interest rate equal to 0.04.

At time $-4$, you deposit an additional amount $3A$ into the account and the continuously compounded risk-free interest rate changes to 0.06.

Which of the following best describes your balance at time 8?

- (a) $A(e^{0.16} + 3e^{0.24})$
- (b) $A(e^{0.32} + 3e^{0.24})$
- (c) $A(e^{0.40} + 3e^{0.24})$
- (d) $A(e^{0.40} + 3e^{0.48})$
- (e) None of the above

**Solution:** (c)

The balance is

$$Ae^{0.04 \cdot 4 + 0.06 \cdot 4} + 3Ae^{0.06 \cdot 4}.$$

**Problem 1.2.** (5 pts) Roger initially deposits $4,000 in an investment fund which pays him $2,000 at time 1 and $4,000 at time 2.

Sally gets $2,000 at time 0 and $4,000 at time 1, and deposits $5,460 at time 2 in return.

Both investments are governed by compound interest with the same annual effective interest rate $i$ and they have the same net present values.

Find $i$.

- (a) About 9%
- (b) About 10.0%
- (c) About 11.5%
- (d) About 12%
- (e) None of the above

**Solution:** (b)

**Problem 1.3.** (5 pts) Roger makes an initial deposit of $K$ into an account governed by the time-varying continuously compounded risk-free interest rate $r(t) = \frac{9}{10} \sqrt{t}$ (per annum).

At the same time, Harry makes an initial deposit at the same amount into an account governed by the constant annual discount rate $d$.

There are no subsequent deposits to or withdrawals from either of the two accounts.

After 4 years, Roger and Harry realize that the balances in their accounts are equal. Which of the following is the closest to $d$?

- (a) $e^{-6/5}$
- (b) $e^{-1/5}$
- (c) $1 - e^{-1/5}$
- (d) $1 - e^{-6/5}$
- (e) $1$

**Solution:** (d)
Solution: (d)
Without loss of generality, we can set $K = 1$. The balance in Roger’s account at time 4 can be expressed as 
\[ e^{\int_0^4 r(t) \, dt} = e^{\frac{9}{10} \cdot \frac{2}{3}^3} = e^{24/5}. \]
So, the balance in Harry’s account is 
\[ e^{24/5} = (1 - d)^{-4} \quad \Rightarrow \quad d = 1 - e^{-6/5}. \]
Please provide your complete solution to the following problems.

**Problem 1.4.** (10 points) By scenario A there is an offer to pay at the rate of $10,000 per annum, continuously, for the next 10 years. By scenario B it is offered to pay the amount $X$ at the end of each of the next 10 years. The force of interest applying to both scenarios is 12%. Find the value of $X$ such that you are indifferent between these two scenarios in the sense that they have the same present values.

**Solution:** The equality of present values of the two annuities translates into

$$10000\overline{a}_{\bar{m}}^{\delta=0.12} = X\overline{a}_{\bar{m}}^{\delta=0.12} \Rightarrow 10000 \frac{1 - e^{-1.2}}{0.12} = X \frac{1 - e^{-1.2}}{e^{1.2} - 1}.$$ 

So,

$$X = \frac{10000(e^{1.2} - 1)}{0.12} \approx 10624.74.$$ 

**Problem 1.5.** (5 pts) Find the total amount of interest that would be paid on a $1,000 loan over a 10-year period, if the effective interest rate is 0.09 per annum under the following repayment method:

The entire loan plus entire accumulated interest is paid as one lump-sum at the end of the loan term.

**Solution:** Using compound interest, the accumulated value at the end of the 10 years is

$$1000 \cdot 1.09^{10} \approx 2367.36.$$ 

The total amount of interest is

$$2367.36 - 1000 = 1367.36.$$ 

**Problem 1.6.** (2 points) Assume that the force of interest is constant and denoted by $r$. Express the accumulation function $a(t)$ in terms of $r$ for $t \geq 0$.

**Solution:**

$$a(t) = e^{rt}$$

**Example 1.1. A warm-up example**

*Source: “Calculus” by James Stewart.*

“One model of population growth is based on the assumption that the population grows at a rate proportional to the size of the population.” Let us denote the proportionality constant by $k$ and let the function $P(\cdot)$ stand for the size of the population. Then, $P$ must satisfy the following (ordinary differential) equation:

$$\frac{dP(t)}{dt} = kP(t)$$

Let the initial population size be $p_0$. Then, the population size $P(t)$ at time $t \geq 0$ is explicitly given by:

$$P(t) = p_0 e^{kt}$$

Please, provide your complete solution to the following problem:

**Problem 1.7.** (8 points) Continuously compounded interest

Assume that the balance in a savings account is growing so that its rate of growth is proportional to the current balance at any time. Let us denote the proportionality constant by $r$ and let the function $B(\cdot)$ stand for the balance as a function of time. Then, $B$ must satisfy which (ordinary differential) equation?
Solution:

\[
\frac{dB(t)}{dt} = rB(t)
\]

If the initial balance in the account is \( b_0 \), then what is the expression for the balance as a function of time \( t \geq 0 \)?

Solution:

\[
B(t) = b_0 e^{rt}
\]
1.2. **Transaction costs.** Please, read the following lecture note prior to attempting the remaining problems: 

[https://www.ma.utexas.edu/users/mcudina/m339d-lecture-two-transaction-costs.pdf](https://www.ma.utexas.edu/users/mcudina/m339d-lecture-two-transaction-costs.pdf)

Provide your final answer only for the following problems.

**Problem 1.8.** (5 points) What is the cost of purchasing 100 shares of Jiffy, Inc. stock given that the bid-ask prices are $31.25 – $32.00 and that there is a $15.00 commission per transaction?

(a) $1,293  
(b) $3,215  
(c) $3,504  
(d) $3,264  
(e) None of the above.

**Solution:** (b)

\[100 \times 32 + 15 = 3215\]

**Problem 1.9.** (5 points) Source: Prof. Jim Daniel (personal communication).

The bid-ask spread on a share of stock is $98-$102. A 5% commission is paid for either buying or selling. Calculate the round-trip transaction cost.

(a) $14  
(b) $10  
(c) $6  
(d) $4  
(e) None of the above.

**Solution:** (a)

You spend \(102 \times (1 + 0.05) = 107.10\) to buy the asset, and receive \(98 \times (1 - 0.05) = 93.10\) when you sell the asset. The round-trip transaction cost is \(107.10 - 93.10 = 14\).