7.1. **Two binomial periods: American options.** Please, provide your complete solutions to the following problems:

**Problem 7.1.** (10 points) Find the current price of a one-year, $110-strike American put option on a non-dividend-paying stock whose current price is $S(0) = 100$. Assume that the continuously compounded interest rate equals $r = 0.06$.

Use a two-period binomial tree with $u = 1.23$, and $d = 0.86$ to calculate the price $V_P(0)$ of the put option.

**Solution:**

We are pricing a one-year option, so $T = 1$. We are using a two-period tree, so $n = 2$. Hence, $h = T/n = 1/2$. The risk-neutral probability of the stock-price going up is

$$p^* = \frac{e^{0.03} - 0.86}{1.23 - 0.86} \approx 0.46.$$

The put has the following payoffs at the final leaves of the tree provided that it is kept alive throughout the two periods:

$$V_{uu} = 0, V_{ud} = 4.22, V_{dd} = 36.04.$$  

So, the value at the “down” node is

$$V_d = \max[K - S_d, e^{-rh}[p^*V_{ud} + (1 - p^*)V_{dd}]] = \max[110 - 86, e^{-0.03}[0.46 \cdot 4.22 + 0.54 \cdot 36.04]]$$  

$$= \max[24, 20.77] = 24.$$

Note that there is *early exercise* at this node. At the “up” node, there can be no early exercise since the put is out-of-the-money. So,

$$V_u = e^{-rh}[p^*V_{uu} + (1 - p^*)V_{ud}] = e^{-0.03} \cdot 0.54 \cdot 4.22 = 2.21 \cdot 4.22$$

Finally,

$$V_P(0) = \max[110 - 100, e^{-0.03}[0.46 \cdot 2.21 + 0.54 \cdot 24]] = 13.56.$$

**Problem 7.2.** (10 points) Solve problem #49 (p.116) from the Sample MFE Problems.

**Solution:** (b)

**Problem 7.3.** (20 points) Consider a two-period binomial model for a non-dividend paying asset $S$ with $S(0) = 50$ and $u = 1/d = 2$. Let $i = 0.25$ denote the effective interest rate per period. You need to price a European put option on $S$ which expires at the end of the two periods and has the strike $K = 70$.

(i) (10 pts) Find the values of the given option at all the nodes in the binomial tree. In particular, find the no-arbitrage price at time 0 of this option.

(ii) (8 pts) Find the number of shares $\Delta$ one needs to invest in at every node in the tree in order to replicate the option.

(iii) (2 pts) If the option were American, would there be early exercise?

**Solution:** Is is easy to construct the binomial tree with the given parameters and to get the risk-neutral probability $p^* = 1/2$.  

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Since the option is European, we could evaluate its price directly as
\[ V_P(0) = \frac{1}{1.25^2} \cdot \frac{1}{4} \left( (K - S_{uu})^+ + 2(K - S_{ud})^+ + (K - S_{dd})^+ \right) \]
\[ = \frac{1}{1.25^2} \cdot \frac{1}{4} \left( (70 - 200)^+ + 2(70 - 50)^+ + (70 - 12.5)^+ \right) \]
\[ = \frac{4}{25} \cdot 97.5 = 15.6. \]

However, we need to find the value of the put at the other nodes of the tree as well. Evidently, the values at the leaves (notes uu, ud and dd) are precisely the payoffs of the put depending on the stock price, i.e., \( V_{uu} = 0, V_{ud} = 20 \) and \( V_{dd} = 57.5 \).

At node \( u \), we have
\[ V_u = \frac{1}{1.25} \cdot \frac{1}{2} (0 + 20) = 8. \]

At node \( d \), the value is
\[ V_d = \frac{1}{1.25} \cdot \frac{1}{2} (20 + 57.5) = 31. \]

As for the \( \Delta s \),
\[ \Delta_u = \frac{0 - 20}{\frac{3}{2} \cdot 100} = -\frac{2}{15} \]
\[ \Delta_d = \frac{20 - 57.5}{\frac{3}{2} \cdot 25} = -1 \]
\[ \Delta_0 = \frac{8 - 31}{\frac{3}{2} \cdot 50} = -\frac{23}{75} \]

As for early exercise, at node \( d \), the payoff of immediate exercise would be
\[ (70 - 25)^+ = 45 > V_d. \]

So, one would create a higher profit by exercising early.

**Problem 7.4.** (10 points)
Consider a one-period forward binomial tree with \( h = 1, S(0) = 100, r = 0.08, \sigma = 0.3, \delta = 0.08. \)

(a) (5 pts) Find the expression \( V_C^F(0, K) \) for the time–0 price of the American call option on \( S \) with strike \( K \) and maturity at the end of the period.

(b) (3 pts) Determine the condition for the strike \( K \) to be such that early exercise occurs?

(c) (2 pts) In particular, is there early exercise for \( K = 70? \)

**Solution:**
(a) From the given data, we get \( u = e^{0.3} = 1.35 \) and \( d = e^{-0.3} = 0.74. \) So, the risk-neutral probability that the price of the stock rises is \( p^* = \frac{e^{0.08} - 0.08 - 0.74}{1.35 - 0.74} = 0.43. \)

Let us denote the price of the corresponding European call option (as a function of strike \( K \)) by \( V_C(0, K) \). Then,
\[ V_C(0, K) = e^{-0.08} [p^*(S_u - K)^+ + (1 - p^*)(S_d - K)^+] \]
\[ = 0.923[0.43(135 - K)^+ + 0.57(74 - K)^+]. \]

The value of immediate exercise of the American call option at time zero is \( 100 - K. \) Thus, the fair price of the American option is
\[ V_C^A(0, K) = \max[V_C(0, K), 100 - K]. \]

(b) Because of the non-zero interest rate, there is cost to early exercise – we pay the strike before expiration, and lose interest on it.

The decisive condition for early exercise is:
\[ 100 - K > V_C(0, K). \]
Hence, it is necessary that $K < 100$. We differentiate between two cases. In the case that $K \geq 74$, we get
\[
100 - K > 0.923 \cdot 0.43(135 - K) = 53.03 - 0.39K \implies 46.97 > 0.61K \implies 77 > K.
\]
In the case that $K < 74$, we get that the condition for early exercise reads as
\[
7.483 > 0.077K \implies 97 > K.
\]
Hence, early exercise occurs for strikes lower than 77.
(c) Yes.