The binomial asset-pricing model. One-period binomial option pricing. Alternative binomial trees.

6.1. The binomial asset-pricing model. Provide your complete solution for the following problem(s).

Problem 6.1. (10 points) Assume that one of the no-arbitrage conditions in the binomial model for pricing options on a non-dividend paying stock \( S \) is violated. Namely, let

\[ e^{r \cdot h} \leq d < u. \]

Illustrate that the above inequalities indeed violate the no-arbitrage requirement. In other words, construct an arbitrage portfolio and show that your proposed arbitrage portfolio is, indeed, and arbitrage portfolio.

Solution: There are multiple ways to illustrate arbitrage opportunities in the above set-up. We provide just one simple example.

Let today’s stock-price be denoted by \( S(0) \). We simply borrow \( S(0) \) from the money market and buy one share of stock. After one period, according to the binomial model, the stock-price either rises to \( S_u = uS(0) \) or drops to \( S_d = dS(0) \).

Let us denote the value of our portfolio on the second day by \( X_u \) in the case the stock price went up and by \( X_d \) if the stock price went down.

The values of our portfolio in those two cases are

\[ X_u = -e^{r \cdot h} \cdot S(0) + uS(0) > 0 \]
\[ X_d = -e^{r \cdot h} \cdot S(0) + dS(0) \geq 0 \]

We have non-negative payoffs in both cases and a strictly positive payoff in one of the cases. Hence, the above strategy constitutes arbitrage.

6.2. One-period binomial option pricing. Provide your final answer only to the following problem(s):

Problem 6.2. (5 points) Consider a one-period binomial pricing framework, and the following market information:

- The current price of a certain non-dividend-paying stock equals $75.
- For ABC stock, \( u = 1.20 \) and \( d = 0.80 \)
- A one-year default-free zero-coupon bond which can be redeemed for $100 has a price of $94.00

Calculate the risk-neutral probability that the price of ABC stock will go up.

(a) 0.60
(b) 0.62
(c) 0.66
(d) 0.68
(e) None of the above.

Solution: (c)

There are no dividends, so \( \delta = 0 \). Since we are pricing a one-year option in one period, we have \( h = 1 \). A one-year, zero-coupon bond redeemable for $100 costs $94 today, so \( e^r = 100/94 \). Thus,

\[ p^* = \frac{e^{(r-\delta)h-d}}{u-d} = \frac{(100/94) - 0.8}{1.2 - 0.8} \approx 0.66. \]

Problem 6.3. (15 points) Let \( S(0) = $100, K = $105, r = 8%, T = 0.5 \) and \( \delta = 0 \). Suppose that \( u = 1.3 \) and \( d = 0.8 \). Using the one-period binomial model, calculate the following:

a. (5 pts) The fair premium for a European put with the above characteristics.

b. (5 pts) The \( \Delta \) in the corresponding replicating portfolio.

c. (5 pts) The amount \( B \) invested in the riskless asset in the replicating portfolio.
Solution: Note: See Problem 10.1.b. from the McDonald text!
The risk-neutral probability is
\[ p^* = \frac{e^{rT} - d}{u - d} = 0.48. \]

a. We start by drawing the one-period tree per the described model. According to the risk-neutral pricing formula in the binomial model, we have that the time--0 price of the put is
\[ V_P(0) = e^{-rT} \left[ p^* V_u + (1 - p^*) V_d \right] \]
\[ = e^{-0.08 \cdot 0.5} [0.48 \cdot 0 + 0.52 \cdot 25] \]
\[ = 12.5. \]

b. \[ \Delta = \frac{0 - 25}{100(1.3 - 0.8)} = \frac{1}{2}. \]
c. \[ B = V_P(0) - \Delta S(0) = 12.5 + \frac{100}{2} = 62.5. \]

6.3. Alternative binomial trees.

Problem 6.4. Cox-Ross-Rubinstein (CRR)
The Cox-Ross-Rubinstein model is a binomial tree in which the up and down factors are given as
\[ u = e^{\sigma \sqrt{h}}, \quad d = e^{-\sigma \sqrt{h}}, \]
where \( \sigma \) denotes the volatility parameter and \( h \) stands for the length of a single period in a tree.

a. (2 points) What is the ratio \( S_u/S_d \)?
\[ \text{Solution: } S_u/S_d = e^{2\sigma \sqrt{h}}. \]
b. (2 points) What is the (as simplified as possible) expression for the risk-neutral probability of the stock price going up in a single step?
\[ \text{Solution: } p^* = \frac{e^{(r-\delta)h - \sigma \sqrt{h}}}{e^{\sigma \sqrt{h}} - e^{-\sigma \sqrt{h}}} = \frac{e^{(r-\delta)h + \sigma \sqrt{h}} - 1}{e^{2\sigma \sqrt{h}} - 1} \]
Substantial further simplification is impossible.
c. (2 points) Express \( S_{ud} \) in terms of \( S(0), \sigma \) and \( h \) in a CRR tree.
\[ \text{Solution: } S_{ud} = S(0) \]
d. (5 points) As was the case with the forward tree, the no-arbitrage condition for the binomial asset-pricing model is satisfied for the CRR tree regardless of the specific values of \( \sigma, \delta, r \) and \( h \). True or false?
\[ \text{Solution: } \text{FALSE} \]

Problem 6.5. The Jarrow-Rudd model.
The Jarrow-Rudd model (aka, the lognormal binomial tree) is a binomial tree in which the up and down factors are defined as follows
\[ u = e^{(r-\delta - \frac{\sigma^2}{2})h + \sigma \sqrt{h}}, \quad d = e^{(r-\delta - \frac{\sigma^2}{2})h - \sigma \sqrt{h}}, \]
where
- \( r \) stands for the continuously-compounded, risk-free interest rate,
- \( \delta \) is the stock’s dividend yield,
- \( \sigma \) denotes the volatility parameter, and
- \( h \) stands for the length of a single period in a tree.

Answer the following questions:

a. (2 points) What is the ratio \( S_u/S_d \)?
\[ \text{Solution: } S_u/S_d = e^{2\sigma \sqrt{h}}. \]
b. (2 points) What is the (as simplified as possible) expression for the risk-neutral probability of the stock price going up in a single step?

Solution:

\[ p^* = \frac{e^{(r-\delta)h - \frac{\sigma^2}{2}h-\sigma\sqrt{h}}}{e^{(r-\delta-\frac{\sigma^2}{2})h+\sigma\sqrt{h}} - e^{(r-\delta-\frac{\sigma^2}{2})h-\sigma\sqrt{h}}} = \frac{1 - e^{-\frac{\sigma^2}{2}h-\sigma\sqrt{h}}}{e^{-\frac{\sigma^2}{2}h+\sigma\sqrt{h}} - e^{-\frac{\sigma^2}{2}h-\sigma\sqrt{h}}} \]

Substantial further simplification is impossible.

c. (5 points) As was the case with the forward tree, the no-arbitrage condition for the binomial asset-pricing model is satisfied for the Jarrow-Rudd tree regardless of the specific values of \( \sigma, \delta, r \) and \( h \).

True or false?

Solution:  **FALSE**