18.1. **Introduction.** Now that we have learned how to use the one-period binomial asset-pricing model to price European-style derivative securities, the question arises on how to set up the asset-pricing model itself. More precisely, how do we define \( u \) and \( d \)?

We still model a market in which there is the possibility of a riskless investment. The interest rate governing the riskless investment is the continuously-compounded, risk-free interest rate \( r \).

For now, the risky asset is a continuous-dividend-paying stock \( S \) with the dividend yield equal to \( \delta \). In the future, we will be able to handle similar risky assets – such as foreign currencies, market indices, or futures contracts – in an analogous way. The new parameter we introduce is a measure of the variability of stock prices. More precisely, it is understood as the annualized standard deviation of the realized returns on the stock. We call it the **volatility**, and denote it by \( \sigma \). It is customary to assume that \( \sigma > 0 \).

The length of a single period in our binomial tree is still equal to \( h \). However, the volatility parameter above corresponds to the length of time equal to one year. We assume that realized returns are

- identically distributed for time periods of the same length (this is *time homogeneity*), and
- *independent* over disjoint time intervals (or time intervals only touching at an endpoint).

It is straightforward, then, to show that the appropriate rescaling of the volatility parameter to a time period of length \( h \) is

\[
\sigma_h = \sigma \sqrt{h}.
\]

18.2. **The forward tree definition.** Consider the forward contract of stock \( S \) for delivery at time \( h \).

**Question 18.1.** What is the **forward price** for delivery of one share of \( S \) at time \( h \)?

**Solution:**

\[
F_{0,h}(S) = S(0)e^{(r-\delta)h}
\]

The \( S_u \) and \( S_d \) in the forward tree are modeled so that the return of the forward contract is, in a sense, centered between the returns for the “up” and “down” states-of-the-world. We set

\[
S_u = F_{0,h}(S)e^{\sigma \sqrt{h}} = S(0)e^{(r-\delta)h+\sigma \sqrt{h}},
\]

\[
S_d = F_{0,h}(S)e^{-\sigma \sqrt{h}} = S(0)e^{(r-\delta)h-\sigma \sqrt{h}}.
\]

In other words, \( u \) and \( d \) are explicitly given by

\[
u = e^{(r-\delta)h+\sigma \sqrt{h}} \quad \text{and} \quad d = e^{(r-\delta)h-\sigma \sqrt{h}}.
\]
Question 18.2. What is the ratio $S_u/S_d$?

Solution: 

$$\frac{S_u}{S_d} = e^{2\sigma\sqrt{h}}$$

Question 18.3. What additional conditions need to be made on $u$ and $d$ so that the no-arbitrage condition for the binomial asset pricing model is

Solution: None, since

$$d < e^{(r-\Delta t)h} < u$$

is equivalent to

$$e^{-\sigma\sqrt{h}} < 1 < e^{\sigma\sqrt{h}}$$

and true for every $\sigma > 0$.

18.3. The risk-neutral probability.

Question 18.4. What is the expression for the risk-neutral probability in the forward tree?

Solution:

$$p^* = \frac{e^{(r-\delta)h} - d}{u - d} = \cdots = \frac{1}{1 + e^{\sigma\sqrt{h}}}.$$ 

Question 18.5. What is the limit of $p^*$ as $h \to 0$?

Solution: $1/2$

18.4. Exercises.

Problem 18.1. Consider a non-dividend-paying stock with a current price of $70 per share. Its volatility is given to be 0.25.

The continuously-compounded, risk-free interest rate equals 4%.

We use a one-period forward binomial tree to model the stock price at the end of the one year.

What is the price of a one-year, at-the-money European call option on this stock consistent with the above stock-price model?

Solution: The up and down factors are 

$$u = e^{0.04+0.25} = 1.3364, \quad d = e^{-0.21} < 1.$$ 

The risk-neutral probability equals 

$$p^* = \frac{1}{1 + e^{0.25}} = 0.4378.$$ 

So, 

$$V_C(0) = e^{-0.04} \times 0.4378 \times 70 \times (1.3364 - 1) = 9.9056.$$