Properties of European put/call prices.

D: March 27th, 2019.

No Arbitrage

⇓

Law of the Unique Price

⇙ ⇘

- **Forwards on stocks:**

$$F^{P}_{0,T}(S) = \begin{cases} S(0), & \text{no div} \\ S(0)e^{-\delta \cdot T}, & \text{cont. div.} \\ S(0) - PV(Div), & \text{discrete} \end{cases}$$

- **Put-call Parity:**

$$V_C(0) - V_P(0) = F^{P}_{0,T}(S) - PV_{0,T}(K)$$

EQUALITIES

Inequalities involving call/put prices:

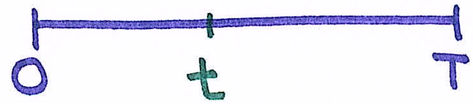
→ **bounds**

→ monotonicity w.r.t. K

→ convexity w.r.t. K

Bounds on Call/Put Prices

T... exercise date →



K... strike price

- $S(t)$ ... time-t stock price
- $V_C(t)$ ... time-t call price
- $V_P(t)$ ... time-t put price

CALLS. Lower bounds: $V_C(t) \geq 0$

Using put-call parity:

$$V_C(t) \geq \underbrace{V_C(t) - V_P(t)}_{\geq 0 \text{ (under } V_P(t))} = F^P_{t,T}(S) - PV_{t,T}(K)$$

$$\Rightarrow \boxed{V_C(t) \geq MAX\left[F^P_{t,T}(S) - PV_{t,T}(K), 0\right]}$$

Q: What if the above inequality is observed not to hold in a certain market (model)?

→: Case #1. $V_C(t) < 0 \Rightarrow$ long the call

Case #2. $V_C(t) < F^P_{t,T}(S) - PV_{t,T}(K)$

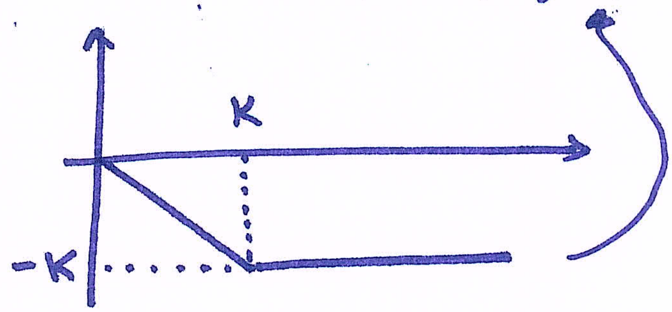
Propose to:
- long the call
- short the prepaid forward

$\Rightarrow$ Cost @ time-t:

$$V_C(t) - F^P_{t,T}(S) < -PV_{t,T}(K)$$

Payoff: $(S(T)-K)_+ - S(T) =$

$= -\min(S(T), K) \geq -K$

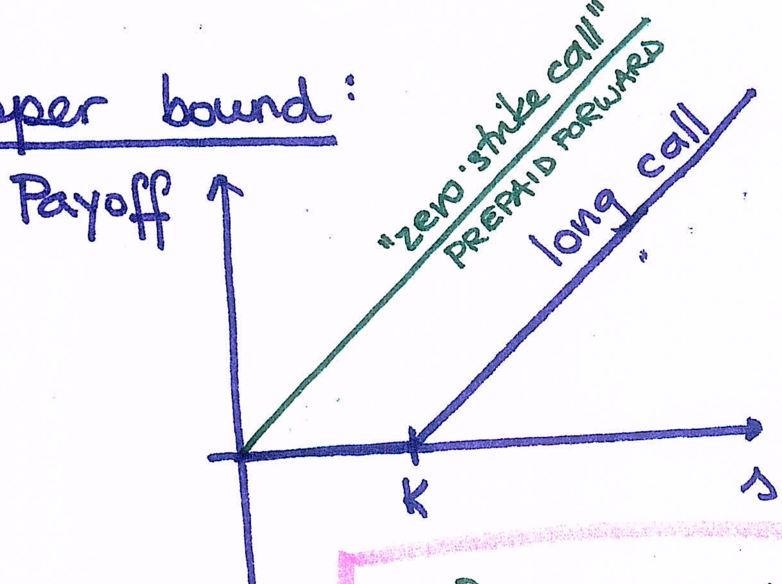


$\Rightarrow$ Profit $=$ Payoff $- FV_{t,T}(\text{Cost @ time } t)$

$= -\min(S(T), K) - FV_{t,T}(\text{Cost @ time } t)$

$> -K - FV(-PV(K)) = 0$ ∎

Upper bound:



The payoff of the prepaid forward dominates the payoff of the long call.

$$\Rightarrow \boxed{F^P_{t,T}(S) \geq V_C(t)}$$

PUTS. Lower bound: $V_P(t) \geq 0$

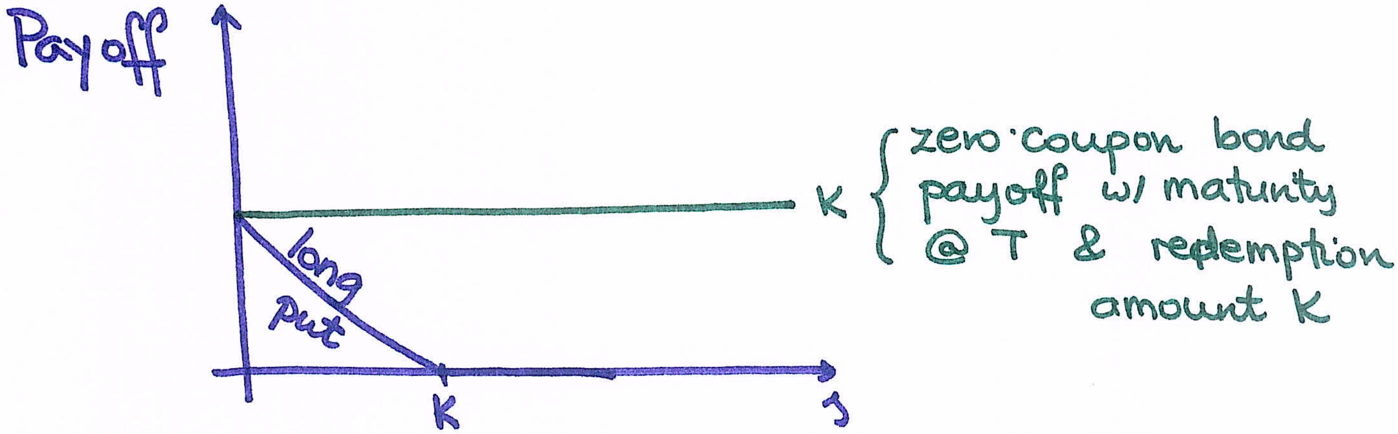
Using put-call parity:

$$V_P(t) \geq V_P(t) - \underbrace{V_C(t)}_{\geq 0} = -F^P_{t,T}(S) + PV_{t,T}(K)$$

$\Rightarrow$

$$V_P(t) \geq \text{MAX}[0; PV_{t,T}(K) - F^P_{t,T}(S)]$$

Upper bound.



$$\Rightarrow \quad PV_{t,T}(K) \geq V_P(t)$$

Q: Let $T \to \infty$. What happens w/ the call/put prices in the limit?

- Assuming $r > 0 \Rightarrow V_P(t, T) \xrightarrow{T \to \infty} 0$ (T: exercise date)

- Assuming $\delta > 0$.

$$S(t)e^{-\delta(T-t)} \geq V_C(t,T) \geq 0$$

$$S(t)e^{-\delta(T-t)} \xrightarrow{T\to\infty} 0, \qquad V_C(t,T) \xrightarrow{T\to\infty} 0$$

Call / Put prices as functions of the strike K.

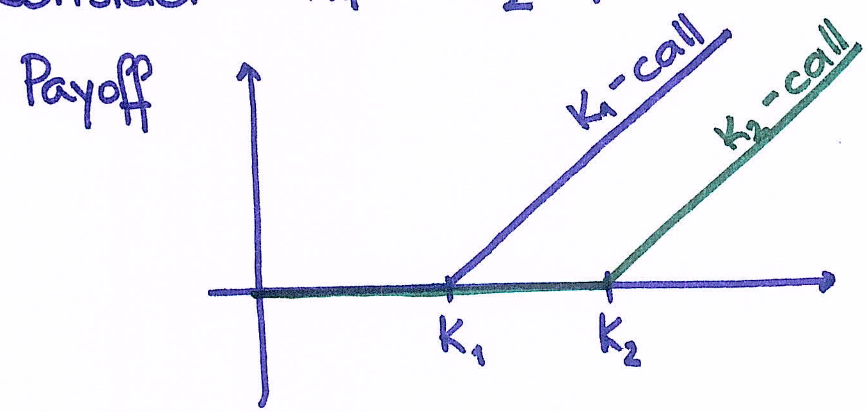
Assume that all other inputs are fixed except for the strike K.
Set the valuation date $t = 0$ and omit it from the notation.
Allow for K to vary:

$V_C(K)$ ... call
$V_P(K)$ ... put
} price w/ strike K

Monotonicity.

Start w/ call prices.

Consider $K_1 < K_2$.



Notice: The payoff of the K_1-strike call dominates the payoff of the K_2-strike call.

$\Rightarrow$ $V_C(K_1) \geq V_C(K_2)$

European call prices are **DECREASING** w/ respect to the strike price.