Call · Price Monotonicity [cont'd]

D: March 29^{th}, 2019.

Recall : European call prices are decreasing as functions of the strike, i.e., for $K_1 < K_2$ we have $V_c(K_1) \geq V_c(K_2)$.

→: Assume, to the contrary, that there exist $K_1 < K_2$ such that $V_c(K_1) < V_c(K_2)$.

I. Suspicion ✓

II. Propose an arbitrage portfolio :

$$\left\{ \begin{array}{l} \cdot \text{ long the } K_1\text{-call} \\ \cdot \text{ write the } K_2\text{-call} \end{array} \right\}$$

CALL BULL SPREAD

III. Verification.

Init. Cost : $V_c(K_1) - V_c(K_2) < 0$

Payoff :

$$(S(T) - K_1)_+ - (S(T) - K_2)_+ =$$

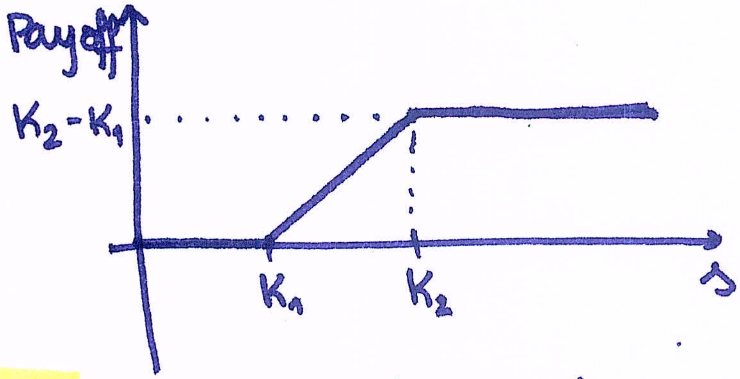
$$= \begin{cases} 0, & \text{if } S(T) < K_1 \\ S(T) - K_1, & \text{if } K_1 \leq S(T) < K_2 \\ \cancel{S(T)} - K_1 - \cancel{S(T)} + K_2 = K_2 - K_1, & \text{if } S(T) \geq K_2 \end{cases}$$

$\Rightarrow$ Payoff ≥ 0

$\Rightarrow$ Profit > 0

$\Rightarrow$ This is, indeed, an arbitrage port. ■

Note:

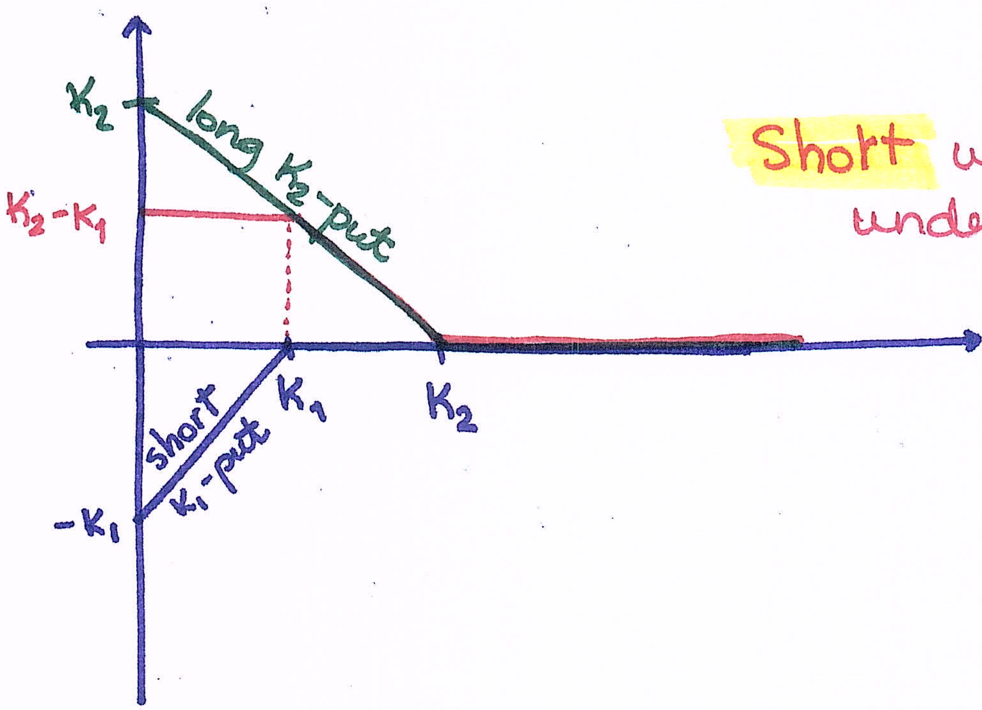


Long w/ respect to the underlying
$\Rightarrow$ it's a suitable hedge for a short position.

Q: How would one construct a PUT BULL SPREAD, i.e., a position consisting of K_1- and K_2-strike puts w/ the same payoff shape?

An idea:
- write? the K_1-put
- long? the K_2-put

PUT **BEAR** SPREAD



Short w.r.t the underlying

Actually, the PUT BULL SPREAD:

$\begin{cases} \bullet \text{ a LONG } K_1\text{-put} \\ \bullet \text{ a WRITTEN } K_2\text{-put} \end{cases}$

Q: What's the difference between the profit of the call bull spread & the put bull spread?

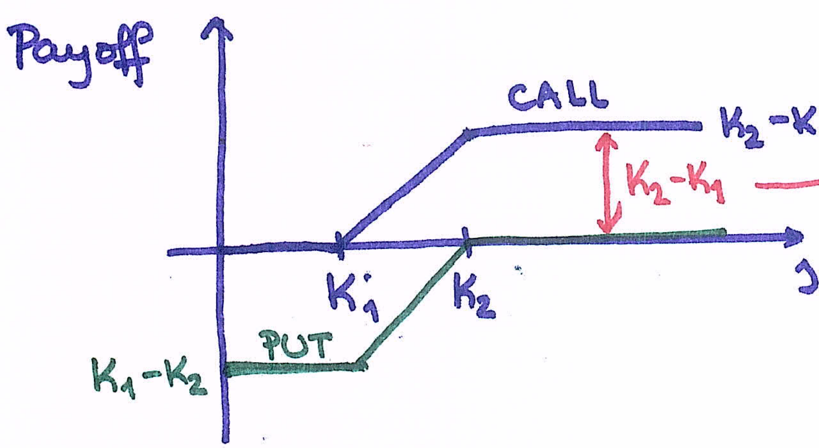
- Init. Cost(Call Bull Sp.) - Init. Cost (Put Bull Sp.) =

$$= \left(\underline{V_c(K_1)} - \underline{V_c(K_2)}\right) - \left(\underline{V_p(K_1)} - \underline{V_p(K_2)}\right) \quad \boxed{\text{Put-Call Parity}}$$

$$= \cancel{F^P_{0,T}}(S) - PV_{0,T}(K_1) - \left(\cancel{F^P_{0,T}}(S) - PV_{0,T}(K_2)\right)$$

$$= PV_{0,T}(K_2 - K_1)$$

- Payoff (Call Bull Sp) - Payoff (Put Bull Sp) = $K_2 - K_1$



$\Rightarrow$ The profits of the call bull spread & the put bull spread are identical!

3.

<u>Claim</u>: Put prices are <u>increasing</u> as functions of the strike, i.e., for every $K_1 < K_2$, we have $V_P(K_1) \leq V_P(K_2)$.

→: Assume, to the contrary, that there exist $K_1 < K_2$ such that $V_P(K_1) > V_P(K_2)$.

We propose the (K_1, K_2) - PUT BEAR SPREAD as an arbitrage portfolio.

To <u>verify</u>: $\begin{cases} \cdot \text{ Init Cost} = V_P(K_2) - V_P(K_1) < 0 \\ \cdot \text{ Payoff} \geq 0 \end{cases}$

↓

Profit > 0 ⇒ This is <u>indeed</u> an arbitrage portfolio!

Cord·Slope Bounds.

Let $K_1 < K_2$

$$0 \leq \begin{Bmatrix} V_C(K_1) - V_C(K_2) \\ V_P(K_2) - V_P(K_1) \end{Bmatrix} \leq PV_{0,T}(K_2 - K_1)$$

MONOTONICITY!

Claim.

$\boxed{\text{CALLS.}}$ Assume, to the contrary, that there exist $K_1 < K_2$ such that

$$V_C(K_1) - V_C(K_2) > PV_{0,T}(K_2 - K_1)$$

$$\Longleftrightarrow \quad V_C(K_1) > V_C(K_2) + PV_{0,T}(K_2 - K_1)$$

I. Suspicion. ✓

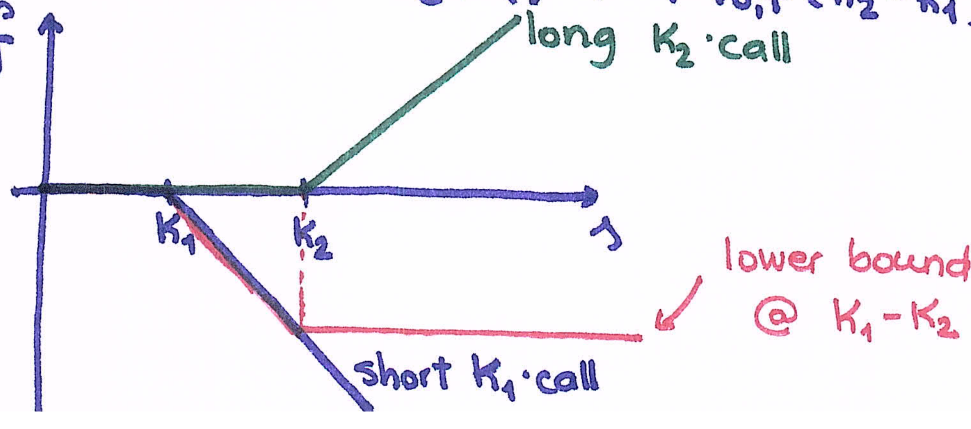
II. Propose an arbitrage portfolio:

$\left\{\begin{array}{l} \cdot \text{ \underline{write} the } K_1\text{-call} \\ \cdot \text{ \underline{long} the } K_2\text{-call} \end{array}\right\}$ CALL BEAR SPREAD ☺

III. Verification.

Init. Cost: $V_C(K_2) - V_C(K_1) < -PV_{0,T}(K_2 - K_1)$

Payoff



$\Rightarrow$ Profit $> \cancel{K_1} - \cancel{K_2} + FV(+PV(\cancel{K_2} - \cancel{K_1})) = 0$

$\Rightarrow$ we did construct an arbitrage portfolio!

$\boxed{\text{PUTS.}}$ Assume, to the contrary, that there exist $K_1 < K_2$ such that

$$V_P(K_2) - V_P(K_1) > PV_{0,T}(K_2 - K_1).$$

We propose the PUT BULL Spread as an arbitrage portfolio. It works out! ∎

12. You are given:

(i) $C(K, T)$ denotes the current price of a K-strike T-year European call option on a nondividend-paying stock.

(ii) $P(K, T)$ denotes the current price of a K-strike T-year European put option on the same stock.

(iii) S denotes the current price of the stock.

(iv) The continuously compounded risk-free interest rate is r.

Which of the following is (are) correct?

MONOTONICITY!

(I) $0 \leq C(50, T) - C(55, T) \leq 5e^{-rT}$ = PV(55 - 50) True!

(II) $50e^{-rT} \leq P(45, T) - C(50, T) + S \leq 55e^{-rT}$

(III) $45e^{-rT} \leq P(45, T) - C(50, T) + S \leq 50e^{-rT}$

(A) (I) only

X (B) (II) only

X (C) (III) only

(D) (I) and (II) only

(E) (I) and (III) only

Focus on:

$$\underbrace{P(45,T)}_{\text{Put-call Parity}} - C(50,T) + S$$

$$\underbrace{\left(C(45,T) - \cancel{F^P_{0,T}(S)} + 45e^{-r\cdot T}\right)}$$

$-C(50,T)$ $\cancel{+S}$ no dividends

$$= \underbrace{C(45,T) - C(50,T)}_{0 \leq \;\cdot\; \leq 5e^{-r\cdot T}} + 45e^{-rT}$$

$45e^{-r\cdot T} \leq \cdot \leq 50e^{-r\cdot T}$

$\Rightarrow$ (III) TRUE! $\Rightarrow$ (E)

(7.)