Bounds on European put/call prices

\[ V_c(t), V_p(t) \] .... price of a European \( \begin{cases} \text{CALL} \\ \text{PUT} \end{cases} \) @ time-\( t \)

w/ exercise date \( T \),
strike price \( K \),
and \( S(t) \) denoting the price of
the underlying @ time-\( t \).

**Calls.**

\[ ?? \geq V_c(t) \geq ?? \]

\[ \text{upper bound} \quad \text{lower bound} \]

**Upper bound.**

\[ \Rightarrow \quad F_{t,T}^p(S) \geq V_c(t) \]

**Lower bound.**

\[ V_c(t) \geq 0 \]

* Put-call parity:

\[ V_c(t) \geq V_c(t) - V_p(t) = F_{t,T}^p(S) - PV_{t,T}(K) \]

\[ \geq 0 \]

\[ V_c(t) \geq \max [0, F_{t,T}^p(S) - PV_{t,T}(K)] = (F_{t,T}^p(S) - PV_{t,T}(K))^+ \]

"prepaid forward = zero-strike call"
Q: What if there is a mispricing which violates $\otimes$?

\[ F_{t,T}^P(S) < V_c(t) \]

=>$\quad$ Propose an arbitrage portfolio:

- long the prepaid forward
- write the call option

(Verify @ home ;)

\begin{align*}
\text{Lower bd:} & \quad V_p(t) \geq 0 \\
\text{Put-call parity:} & \quad V_p(t) \geq V_p(t) - V_c(t) = PV_{t,T}(K) - F_{t,T}^P(S) \\
& \quad \Rightarrow V_p(t) \geq (PV_{t,T}(K) - F_{t,T}^P(S))_+
\end{align*}

\begin{itemize}
  \item Upper bd:
  \text{Payoff:}
  \begin{align*}
  & \text{K zero-coupon bonds w/ maturity } T \\
  & V_p(t) \leq PV_{t,T}(K)
  \end{align*}
\end{itemize}
European-call/put prices as functions of the strike

From now on, all options are valued at time-0.

\[ V_c(K) \] ... call \[ V_p(K) \] ... put \} prices with all other inputs fixed.

Monotonicity.

A function \( f \) is:

- increasing/nondecreasing if \( x_1 < x_2 \) then \( f(x_1) \leq f(x_2) \)
- decreasing/nonincreasing if \( x_1 < x_2 \) then \( f(x_1) \geq f(x_2) \)

Calls. Let \( K_1 < K_2 \).

Q: \[ V_c(K_1) \ ? \ V_c(K_2) \]

The payoff of the \( K_1 \)-strike call dominates the payoff of the \( K_2 \)-strike call.

\[ \Rightarrow V_c(K_1) \geq V_c(K_2) \]

Call prices are decreasing with respect to the strike.
Example. Assume, to the contrary, that there exists a pair \( K_1 < K_2 \) such that 
\[
V_c(K_1) < V_c(K_2)
\]

I. Suspicion.

II. Construction.

Propose an arbitrage portfolio:
- write the \( K_2 \)-strike call
- buy the \( K_1 \)-strike call

III. Verification.

Initial Cost: 
\[
V_c(K_1) - V_c(K_2) < 0
\]

\( \Downarrow \)

Initial inflow of cash.

Payoff:
\[
(S(T) - K_1)_+ - (S(T) - K_2)_+
\]

\( \Rightarrow \) \( \text{PROFIT} = \text{PAYOFF} - \text{FV(Init.\ Cost)} - \text{FV} (V_c(K_1) - V_c(K_2)) > 0 \)
Q: Note that the above portfolio is long with respect to the underlying. So, it would be an appropriate hedge for a short position.

It is called a (CALL) BULL SPREAD.
Task: Construct a **PUT** bull spread: $K_1 < K_2$

- **long** a $K_1$-strike put 
- **short** a $K_2$-strike put 

to end up w/ a financial position whose payoff curve has _____ shape.

1º Draw the payoff curve
2º What are the bounds?
3º Compare the profits of call and put bull spreads!