Bounds on European-put/call prices

March 24, 2017.

$\left.\begin{array}{l} V_c(t) \\ V_p(t) \end{array}\right\}$ .... price of a European $\left\{\begin{array}{l}\text{CALL}\\ \text{PUT}\end{array}\right\}$ @ time-t

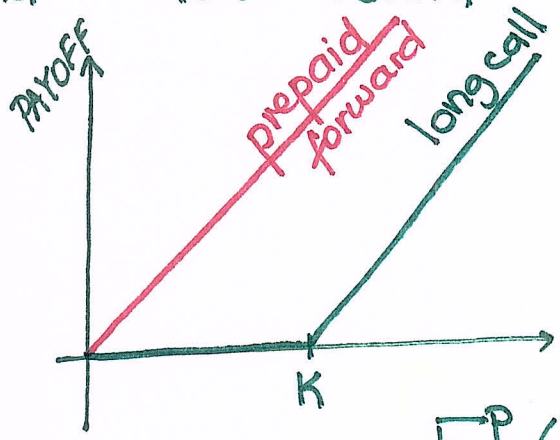
$\uparrow$ valuation date

w/ exercise date T,
strike price K,
and $S(t)$ denoting the price of the underlying @ time-t.

$\boxed{\text{Calls.}}$ $\quad ?? \geq V_c(t) \geq ?$

$\uparrow$ upper bound $\qquad \uparrow$ lower bound

Upper bound.



"prepaid forward = zero-strike call"

$$\Rightarrow \quad F^P_{t,T}(S) \geq V_c(t) \quad \circledast$$

Lower bound.

- $V_c(t) \geq 0$
- Put-call parity:

$$V_c(t) \geq V_c(t) - \underbrace{V_p(t)}_{\geq 0} = F^P_{t,T}(S) - PV_{t,T}(K)$$

$$V_c(t) \geq \text{MAX}\left[0,\ F^P_{t,T}(S) - PV_{t,T}(K)\right]$$
$$= \left(F^P_{t,T}(S) - PV_{t,T}(K)\right)_+$$

(★) Q: What if there is a mispricing which violates (★)?

e.g., $F^P_{t,T}(S) < V_c(t)$

$\Rightarrow$ Propose an arbitrage portfolio:

- <u>long</u> the prepaid forward

and

- <u>write</u> the call option

(Verify @ home :))

Puts.

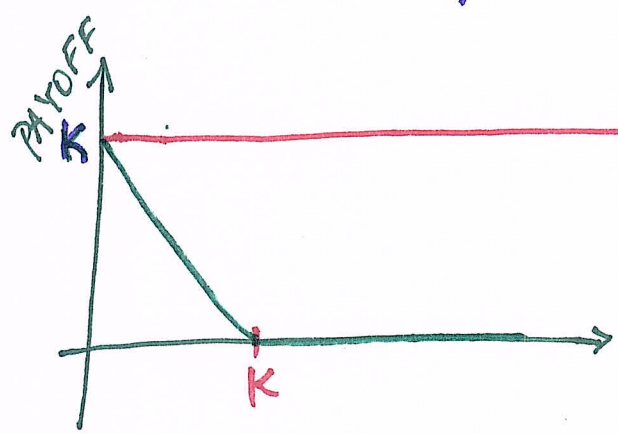
<u>Lower bd:</u>

- $V_p(t) \geq 0$
- <u>Put-call parity:</u>

$$V_p(t) \geq V_p(t) - \underbrace{V_c(t)}_{\geq 0} = PV_{t,T}(K) - F^P_{t,T}(S)$$

$$\Rightarrow V_p(t) \geq \left(PV_{t,T}(K) - F^P_{t,T}(S)\right)_+$$

<u>Upper bd:</u>



K zero-coupon bonds w/ maturity T

$$V_p(t) \leq PV_{t,T}(K)$$

European-call/put prices as functions of the strike

From now on, all options are valuated @time-0.

$V_c(K)$... call
$V_p(K)$... put
} prices with all other inputs fixed.

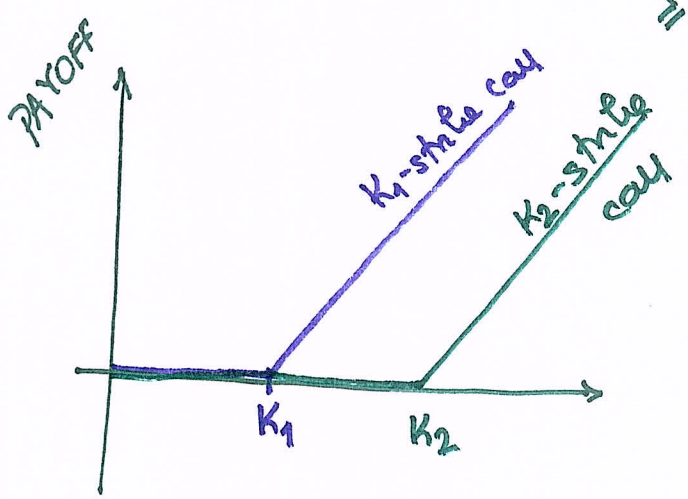
Monotonicity.

A function f is:

- increasing/nondecreasing if $x_1 < x_2$ then $f(x_1) \leq f(x_2)$
- decreasing/nonincreasing if $x_1 < x_2$ then $f(x_1) \geq f(x_2)$

Calls. Let $K_1 < K_2$.

Q: $V_c(K_1) \overset{?}{\geq\leq} V_c(K_2)$



The payoff of the K_1-strike call dominates the payoff of the K_2-strike call.

$\Rightarrow V_c(K_1) \geq V_c(K_2)$

Call prices are decreasing with respect to the strike.

Example. Assume, to the contrary, that there EXISTS a pair $K_1 < K_2$ such that

$$V_c(K_1) < V_c(K_2)$$

I. Suspicion. ✓

II. Construction.

Propose an arbitrage portfolio:

- <u>write</u> the K_2-strike call
- <u>buy</u> the K_1-strike call

III. Verification.

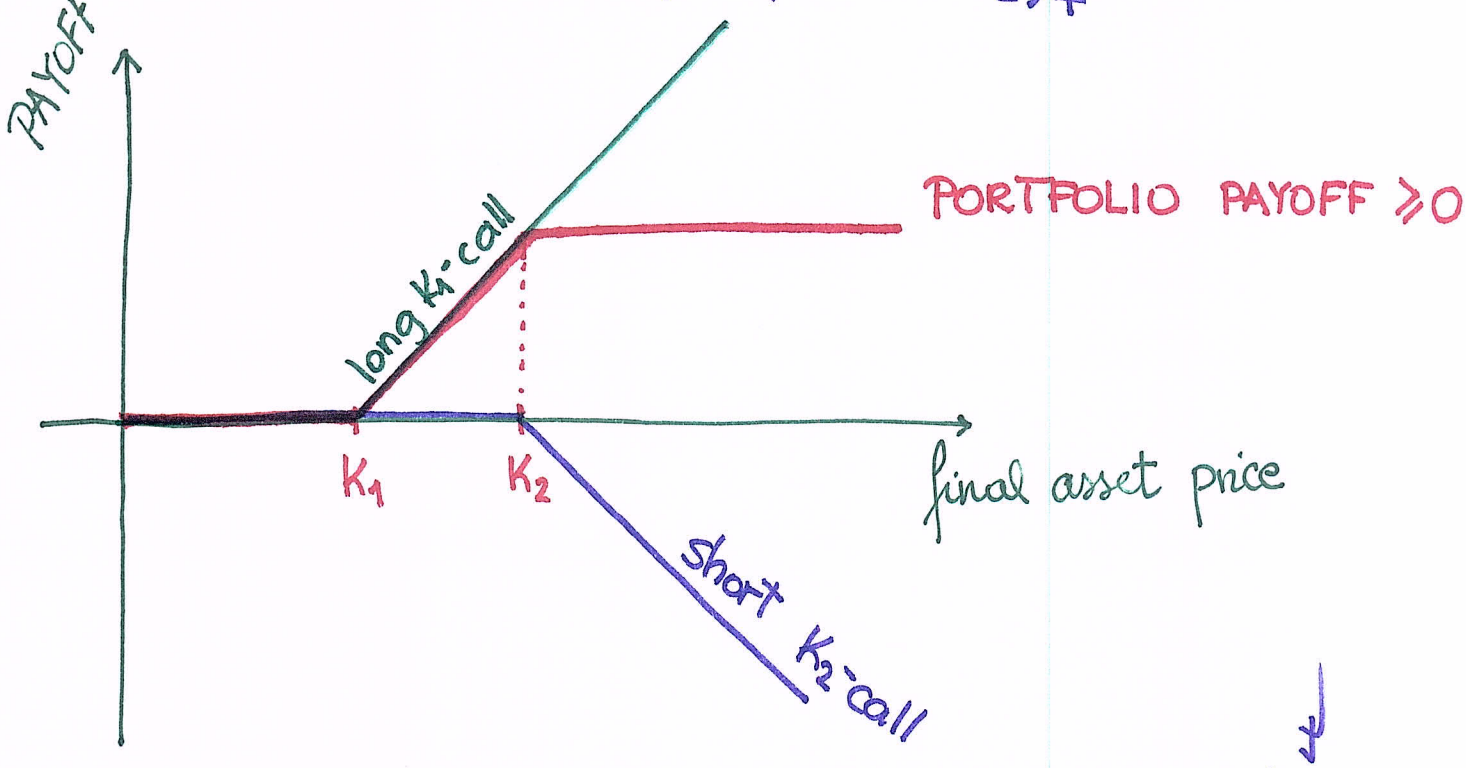
Initial Cost: $V_c(K_1) - V_c(K_2) < 0$

$\Downarrow$

Initial inflow of cash.

<u>Payoff</u>: ?

$$(S(T)-K_1)_+ - (S(T)-K_2)_+$$



$$\Rightarrow \text{PROFIT} = \underbrace{\text{PAYOFF}}_{\geq 0} - FV(\text{Init. Cost}) \geq FV(V_c(K_1) - V_c(K_2)) > 0$$

Q: Note that the above portfolio is <u>Long</u> with respect to the underlying. So, it would be an appropriate hedge for a short position.

It is called a (CALL) BULL SPREAD.

Task: Construct a PUT bull spread: $K_1 < K_2$

- long a K_1-strike put

and

- short a K_2-strike put

to end up w/ a financial position whose payoff curve has  shape.

1° Draw the payoff curve

2° What are the bounds?

3° Compare the profits of call and put bull spreads!