March 27th, 2017.

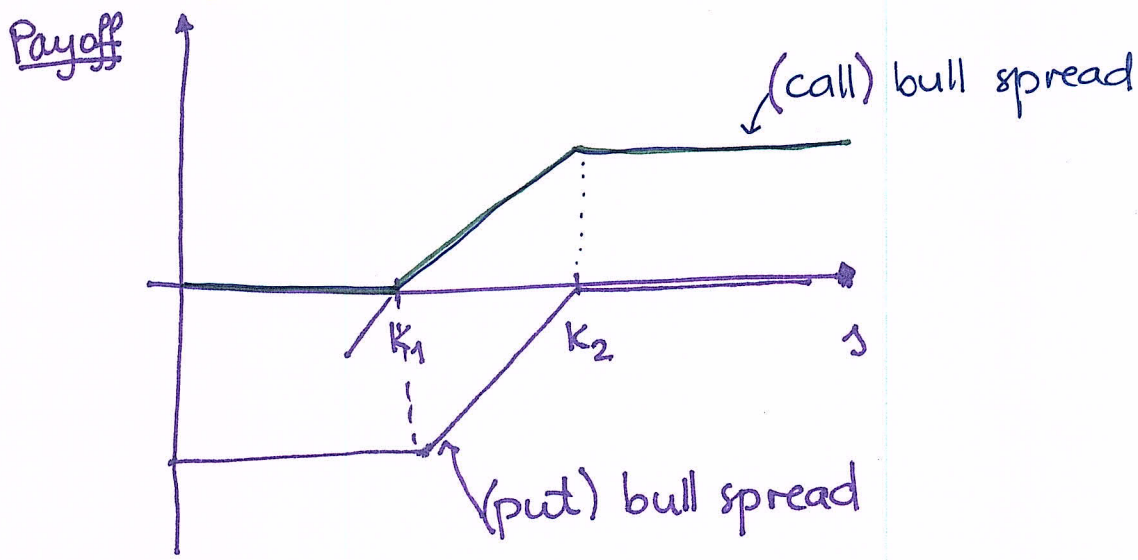
Review: • Bounds on European call/put prices

• Understanding the behavior of option prices as functions of the strike

→ call prices are DECREASING w.r.t. the argument K

* the arbitrage portfolio to use to show this is the call bull spread *

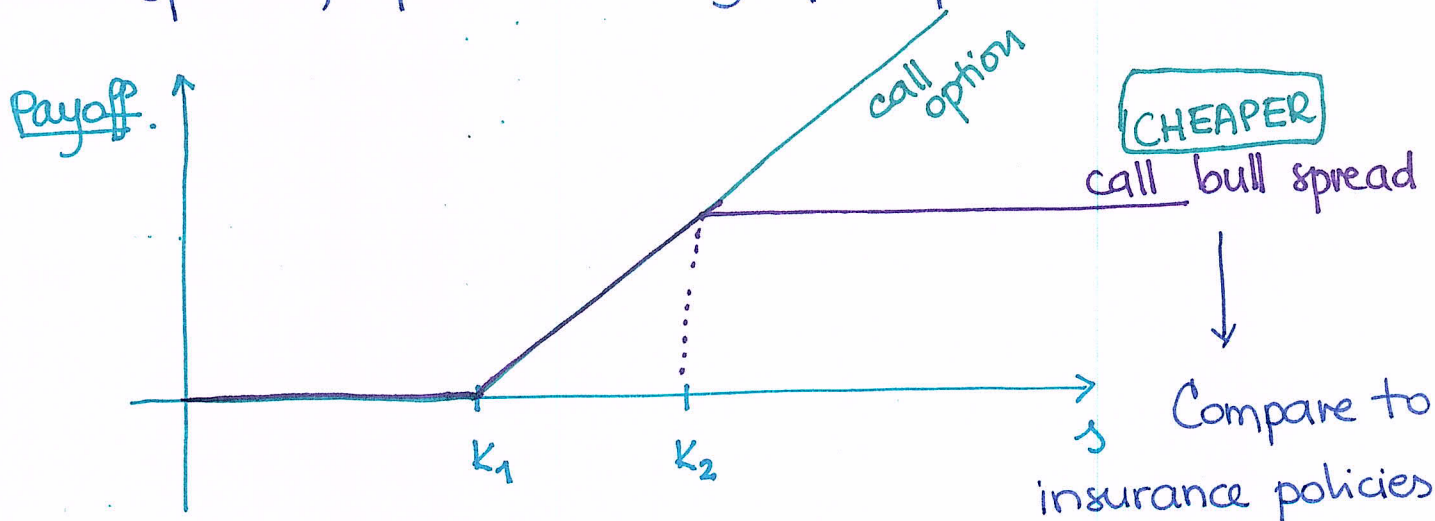
→ The put bull spread



The PROFIT curve is the same for the call bull spread and the put bull spread. So, usually we just say/use $\boxed{\text{bull spreads.}}$

Do this @ home!

Q: What's the "original" use of bull spreads?
Why would an investor be interested in acquiring a bull spread, if not merely for speculation?

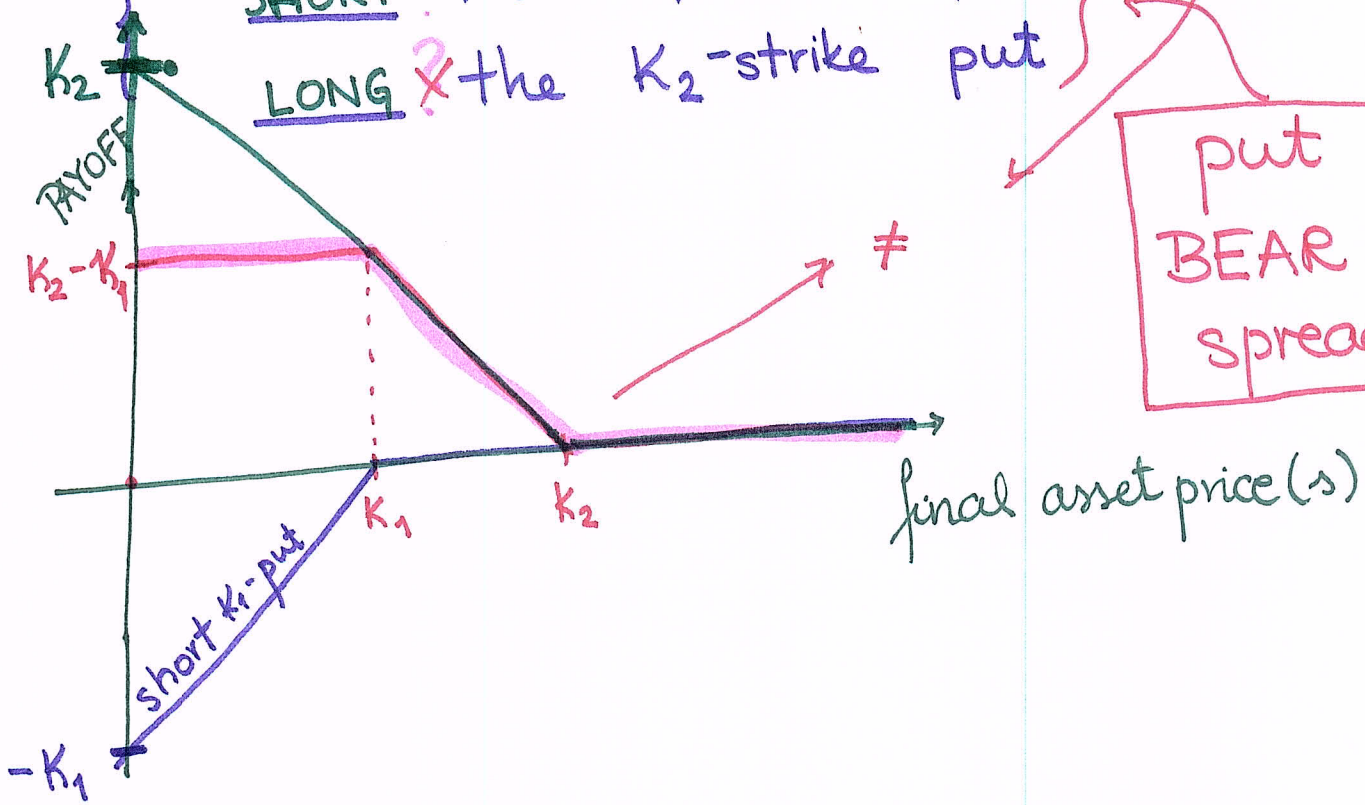


Compare to insurance policies w/ <u>both</u> the deductible and an <u>upper policy limit</u>.

The (put) ~~bull~~ BEAR spread : $K_1 < K_2$

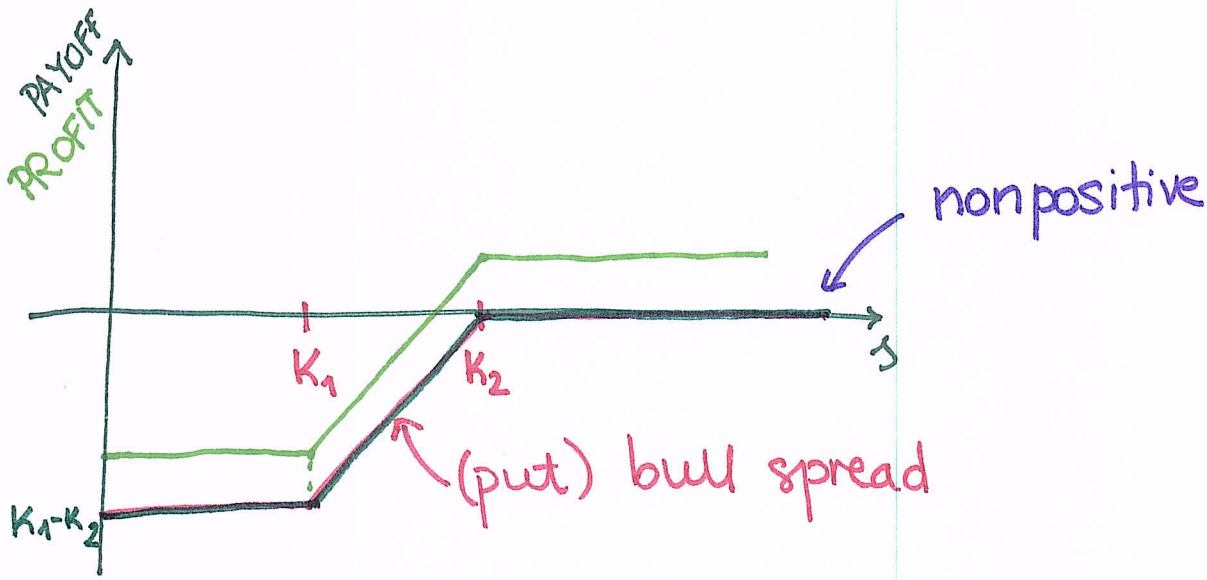
- SHORT ~~X~~ ? the K_1-strike put
- LONG ~~X~~ ? the K_2-strike put

put BEAR spread



The (PUT) BULL SPREAD: $K_1 < K_2$

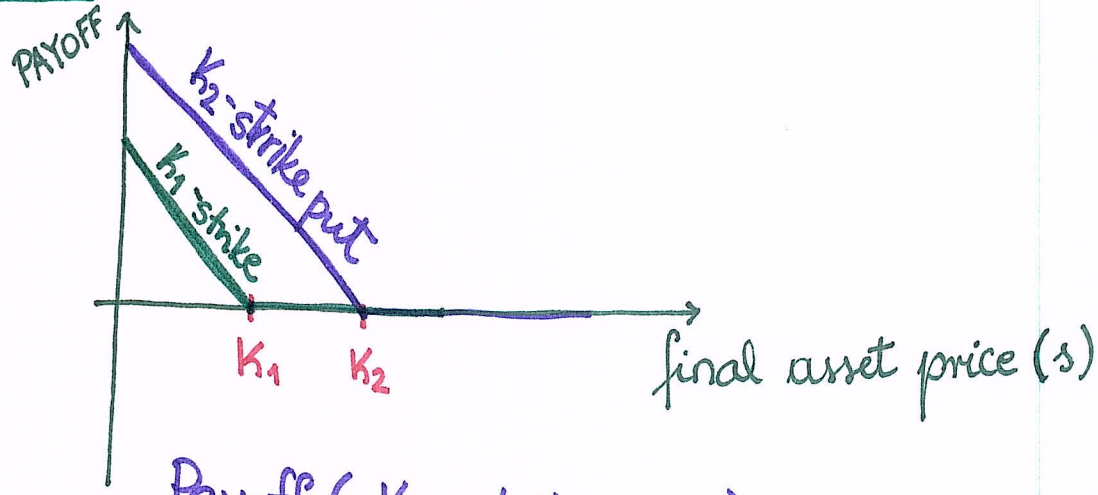
- LONG the K_1-strike put
- SHORT the K_2-strike put



- Compare the profits of the call bull spread & the put bull spread

Monotonicity [cont'd].

PUTS. Let $K_1 < K_2$.



Payoff (K_2-strike put) $\geq$ Payoff (K_1-strike put)

$\Rightarrow$ $V_p(K_1) \leq V_p(K_2)$

In words: European put prices are increasing with respect to the strike.

Example. What if there exist $K_1 < K_2$ such that $V_p(K_1) > V_p(K_2)$?

I. Suspicion. ✓

II. Propose an arbitrage portfolio:

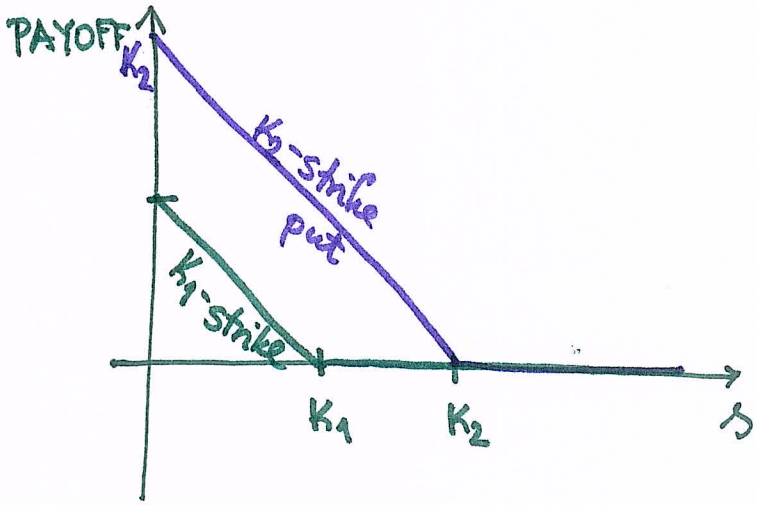
- short the K_1-strike put
- long the K_2-strike put

Recognize: this portfolio is the short version of the put bull spread.

It is actually called: PUT BEAR SPREAD.

Monotonicity [cont'd]

PUTS. Let $K_1 < K_2$.



The two payoff functions are such that:

Payoff (K_2-strike put)

$\geq$

Payoff (K_1-strike put)

$$\Longrightarrow \quad V_P(K_1) \leq V_P(K_2)$$

In words: European put prices are increasing with respect to the strike.

Example. What if there exist $K_1 < K_2$ such that $V_P(K_1) > V_P(K_2)$?

I. Suspect that there's an arbitrage opportunity.

II. Construct a potential arbitrage portfolio:

- write the K_1-strike put
- buy the K_2-strike put

The name of this portfolio: a put bear spread

III. Verification

Initial Cost: $V_p(K_2) - V_p(K_1) < 0$

$\Downarrow$

Initial inflow of cash :)

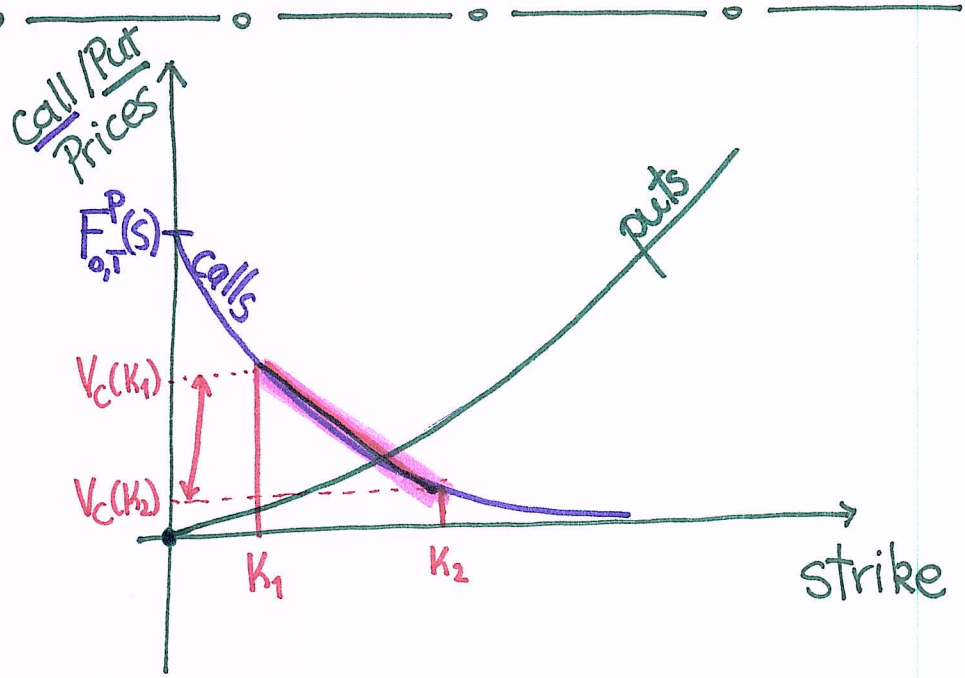
Payoff ≥ 0

$\uparrow$

Earlier today :)

Profit > 0

$\Rightarrow$ ARBITRAGE!



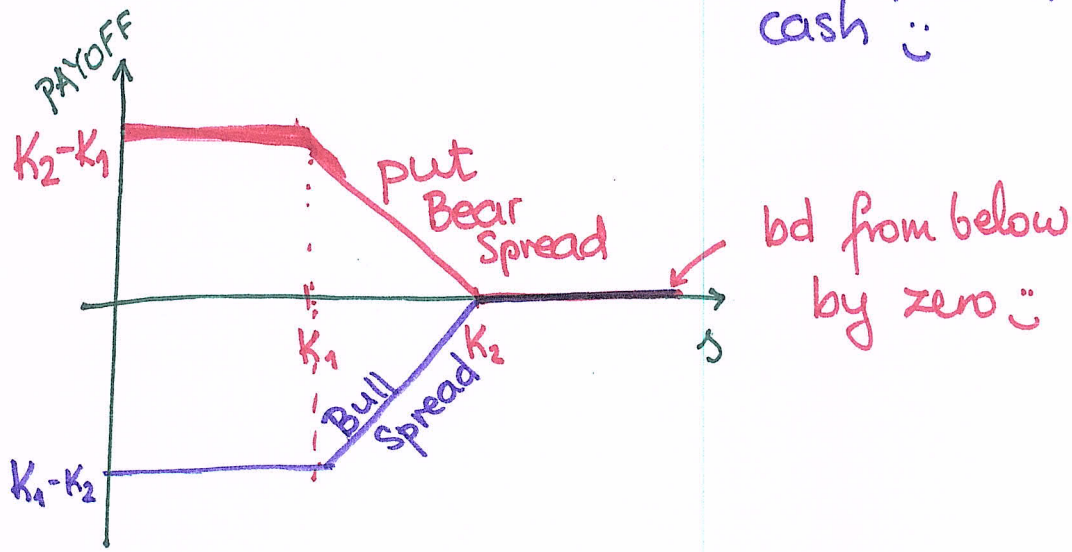
III. Verificiation.

Initial cost: $V_P(K_2) - V_P(K_1) \leq 0$

$\Downarrow$

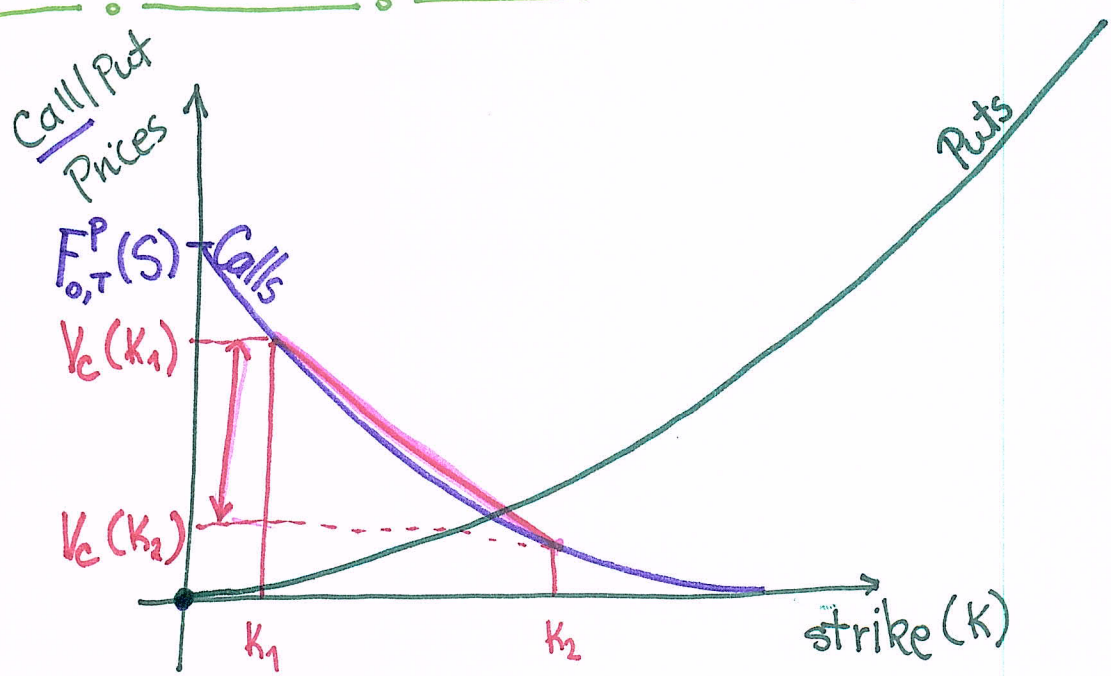
Initial inflow of cash :)

Payoff



Profit > 0

$\Rightarrow$ ARBITRAGE !



Cord-slope inequalities

We already know: $K_1 < K_2$

$$? \geq \begin{cases} V_c(K_1) - V_c(K_2) \\ V_p(K_2) - V_p(K_1) \end{cases} \geq 0$$

Claim: The upper bound on both is $PV_{0,T}(K_2 - K_1)$.

Example. Assume that, to the contrary, there are $K_1 < K_2$ such that

$$V_c(K_1) - V_c(K_2) > PV_{0,T}(K_2 - K_1)$$

I. Suspicion.

II. Construct a potential arbitrage portfolio.

$$V_c(K_1) > V_c(K_2) + PV_{0,T}(K_2 - K_1)$$

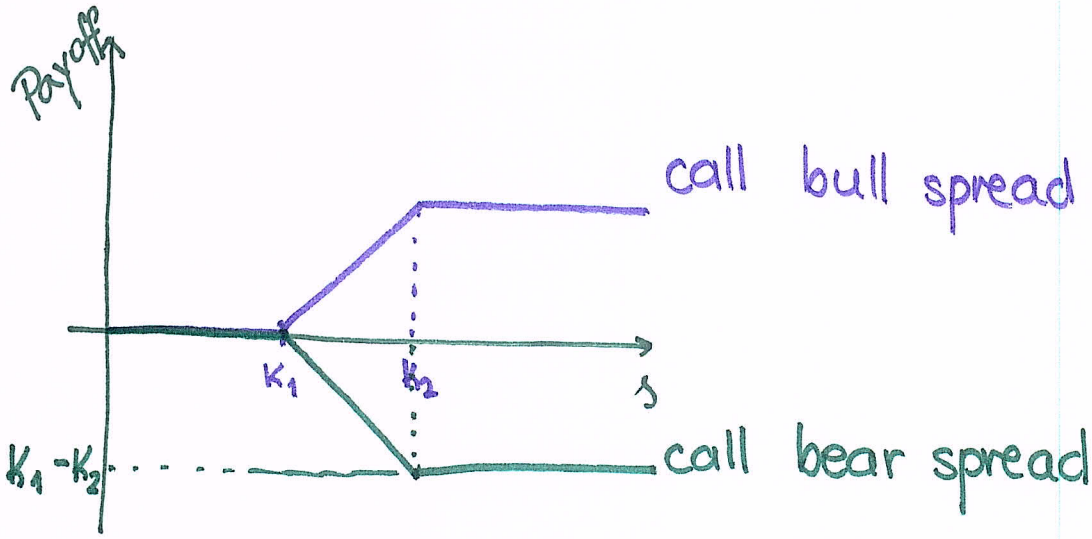
- short the K_1-call
- long the K_2-call

CALL BEAR SPREAD

III. Verification.

Init. Cost: $V_c(K_2) - V_c(K_1)$

Payoff?



$$\Rightarrow \quad \text{Payoff} \geq K_1 - K_2 .$$

$$\text{Profit} = \text{Payoff} - FV_{0,T}\,(\text{Init. Cost})$$

$$\geq K_1 - K_2 + \cancel{FV_{0,T}} \underbrace{\left(-V_c(K_2) + V_c(K_1)\right)}_{\geq \cancel{PV_{0,T}}(K_2 - K_1)}$$

$$\underline{\text{Profit}} > K_1 - K_2 + (K_2 - K_1) \underline{= 0} \quad \Rightarrow \text{ARBITRAGE!}$$

Q: What if the inequality for put prices is violated? More precisely, what if there exist $K_1 < K_2$ such that

$$V_p(K_2) - V_p(K_1) > PV_{0,T}(K_2 - K_1) \; ?$$

Cord-slope inequalities

We already know: IF $K_1 < K_2$, THEN

$$? \geq \left\{ \begin{array}{l} V_c(K_1) - V_c(K_2) \\ V_p(K_2) - V_p(K_1) \end{array} \right\} \geq 0$$

Claim: The upper bound on both is: $PV_{0,T}(K_2 - K_1)$.

Example. Assume that, to the contrary, the is a pair of strikes $K_1 < K_2$ such that

$$V_p(K_2) - V_p(K_1) > PV_{0,T}(K_2 - K_1). \quad *$$

I. Suspect arbitrage. ✓

II. Construct a potential arbitrage portfolio:

- buy K_1-strike put
- write K_2-strike put

} put BULL spread.

III. Verification.

Initial cost: $V_p(K_1) - V_p(K_2)$

Payoff $\geq K_1 - K_2$

$$\text{Profit} = \text{Payoff} - FV_{0,T}(\text{Initial Cost})$$

$$\geq K_1 - K_2 + \cancel{FV_{0,T}}\underbrace{\left(-V_p(K_1) + V_p(K_2)\right)}_{* \;>\; \cancel{PV_{0,T}}(K_2 - K_1)}$$

$$\underline{\text{Profit}} > K_1 - K_2 + (K_2 - K_1) \underline{= 0}$$

$\Rightarrow$ ARBITRAGE!

Q: What if the inequality for call prices is violated? More precisely, what if there exist $K_1 < K_2$ such that

$$V_c(K_1) - V_c(K_2) > PV_{0,T}(K_2 - K_1) \;?$$

12. You are given:

exercise date

(i) $C(K, T)$ denotes the current price of a K-strike T-year European call option on a nondividend-paying stock.

(ii) $P(K, T)$ denotes the current price of a K-strike T-year European put option on the same stock.

(iii) S denotes the current price of the stock.

(iv) The continuously compounded risk-free interest rate is r.

Which of the following is (are) correct?

$PV_{0,T}(55-50) = 5e^{-rT}$

decrease

(I) $0 \le C(50, T) - C(55, T) \le 5e^{-rT}$ $\Rightarrow$ True.

$\Downarrow$

$\cancel{B}$. $\cancel{C}$.

(II) $50e^{-rT} \le P(45, T) - C(50, T) + S \le 55e^{-rT}$

(III) $45e^{-rT} \le P(45, T) - C(50, T) + S \le 50e^{-rT}$

(A) (I) only

(B) (II) only

(C) (III) only

(D) (I) and (II) only

(E) (I) and (III) only

Put-call Parity

$$C(50,T) = P(50,T) + \underbrace{F^P_{0,T}(S)}_{=S} - 50e^{-rT}$$

$$= \underbrace{\underbrace{P(45,T) - P(50,T)}_{\le 0} - \cancel{S} + 50e^{-rT} + \cancel{S}}_{\le 50e^{-rT}}$$

$\Downarrow$

$\cancel{D}$.

$$-5e^{-rT} \cancel{A} \le P(45,T) - P(50,T)$$

$\Downarrow$

(III) is True ☺