- Bounds on European call/put prices
  - Understanding the behavior of option prices as functions of the strike
    - call prices are **DECREASING** w.r.t. the argument $K$
  - the arbitrage portfolio to use to show this is the **call bull spread**

- The put bull spread

The **PROFIT** curve is the same for the call bull spread and the put bull spread. So, usually we just say/use **bull spreads**.

Do this at home!
Q: What's the "original" use of bull spreads? Why would an investor be interested in acquiring a bull spread, if not merely for speculation?

Compare to insurance policies w/ both the deductible and an upper policy limit.
The (put) bull spread: \( K_1 < K_2 \)

- **SHORT** the \( K_1 \)-strike put
- **LONG** the \( K_2 \)-strike put

The (PUT) BULL SPREAD: \( K_1 - K_2 \)

- **LONG** the \( K_1 \)-strike put
- **SHORT** the \( K_2 \)-strike put

Compare the profits of the call bull spread and the put BEAR spread.
Monotonicity [cont'd].

**PUTS.** Let \( K_1 < K_2 \).

\[
\text{Payoff (} K_2 \text{-strike put)} \geq \text{Payoff (} K_1 \text{-strike put)}
\]

\[\Rightarrow V_p(K_1) \leq V_p(K_2)\]

In words: European put prices are increasing with respect to the strike.

**Example.** What if there exist \( K_1 < K_2 \) such that \( V_p(K_1) > V_p(K_2) \)?

I. **Suspicion.** ✓

II. Propose an arbitrage portfolio:
- **short** the \( K_1 \)-strike put
- **long** the \( K_2 \)-strike put

Recognize: this portfolio is the short version of the put bull spread.

It is actually called: **PUT BEAR SPREAD.**
Monotonicity [cont'd]

PUTS.

Let \( K_1 < K_2 \).

The two payoff functions are such that:

Payoff (\( K_2 \)-strike put) \[ V_1 \]

Payoff (\( K_1 \)-strike put)

\[ V_p(K_1) \leq V_p(K_2) \]

In words: European put prices are increasing with respect to the strike.

Example. What if there exist \( K_1 < K_2 \) such that \( V_p(K_1) > V_p(K_2) \) ?

I. Suspect that there's an arbitrage opportunity.

II. Construct a potential arbitrage portfolio:

- write the \( K_1 \)-strike put
- buy the \( K_2 \)-strike put

The name of this portfolio: a put (bear) spread
III. Verification

Initial Cost: \( V_p(K_2) - V_p(K_1) < 0 \)

\[ \downarrow \]

Initial inflow of cash \( \uparrow \)

Payoff \( \geq 0 \)

\[ \uparrow \]

Earlier today \( \downarrow \)

Profit \( > 0 \)

\[ \Rightarrow \text{ARBITRAGE!} \]
Verification.

Initial cost: \( V_p(K_2) - V_p(K_1) \leq 0 \)

Initial inflow of cash \( \uparrow \)

Payoff

\( K_2 - K_1 \)

\( K_1 - K_2 \)

Profit > 0

\( \Rightarrow \) ARBITRAGE \( \uparrow \)
Cord-slope inequalities

We already know: \( K_1 < K_2 \)

\[
\begin{align*}
? & \geq \left\{ V_c(K_1) - V_c(K_2) \right\} \\
& \geq \left\{ V_p(K_2) - V_p(K_1) \right\}
\end{align*}
\]

**Claim:** The upper bound on both is \( PV_{0,T}(K_2 - K_1) \).

**Example.** Assume that, to the contrary, there are \( K_1 < K_2 \) such that

\[ V_c(K_1) - V_c(K_2) > PV_{0,T}(K_2 - K_1) \]

**I. Suspicion.**

**II. Construct a potential arbitrage portfolio.**

\[ V_c(K_1) > V_c(K_2) + PV_{0,T}(K_2 - K_1) \]

- **short** the \( K_1 \)-call
- **long** the \( K_2 \)-call

**III. Verification.**

**Init. Cost:** \( V_c(K_2) - V_c(K_1) \)

**Payoff?**
\[ \text{Profit} = \text{Payoff} - FV_{0,T} \text{ (Init. Cost)} \]
\[ \geq K_1 - K_2 + FV_{0,T} \left( -V_c(K_2) + V_c(K_1) \right) \]
\[ \geq PV_{0,T}(K_2 - K_1) \]

\[ \text{Profit} > K_1 - K_2 + (K_2 - K_1) = 0 \quad \Rightarrow \text{ARBITRAGE!} \]

**Q:** What if the inequality for put prices is violated? More precisely, what if there exist \( K_1 < K_2 \) such that

\[ V_p(K_2) - V_p(K_1) > PV_{0,T}(K_2 - K_1) \]
Card-slope inequalities

We already know: IF \( K_1 < K_2 \), THEN

\[
\{ \begin{align*}
V_c(K_1) - V_c(K_2) \\
V_p(K_2) - V_p(K_1)
\end{align*} \} \geq 0
\]

Claim: The upper bound on both is: \( PV_{0,T}(K_2 - K_1) \).

**Example.** Assume that, to the contrary, there is a pair of strikes \( K_1 < K_2 \) such that

\[ V_p(K_2) - V_p(K_1) > PV_{0,T}(K_2 - K_1). \]

\[ \star \]

I. Suspect arbitrage. \( \checkmark \)

II. Construct a potential arbitrage portfolio:

- buy \( K_1 \)-strike put
- write \( K_2 \)-strike put

\[ \text{put BULL spread.} \]

III. Verification.

Initial cost: \( V_p(K_1) - V_p(K_2) \)

Payoff \( \geq K_1 - K_2 \)
\[
\text{Profit} = \text{Payoff} - FV_{0,T} \text{(Initial Cost)} \\
\geq K_1 - K_2 + FV_{0,T} \left( V_p(K_1) + V_p(K_2) \right) - \underbrace{PV_{0,T}(K_2 - K_1)}_{>}
\]

\[
\text{Profit} > K_1 - K_2 + (K_2 - K_1) = 0
\]

\[
\Rightarrow \text{ARBITRAGE!}
\]

Q: What if the inequality for call prices is violated? More precisely, what if there exist \( k_1 < k_2 \) such that

\[
V_c(k_1) - V_c(k_2) > PV_{0,T}(k_2 - k_1)
\]
12. You are given:

(i) $C(K, T)$ denotes the current price of a $K$-strike $T$-year European call option on a nondividend-paying stock.

(ii) $P(K, T)$ denotes the current price of a $K$-strike $T$-year European put option on the same stock.

(iii) $S$ denotes the current price of the stock.

(iv) The continuously compounded risk-free interest rate is $r$.

Which of the following is (are) correct?

(A) (I) only

(B) (II) only

(C) (III) only

(D) (I) and (II) only

(E) (I) and (III) only

**Put-call Parity**

$C(50, T) = P(50, T) + \frac{F_{0,T}}{e^{-rT}}(S) - 50e^{-rT} = S$

\[
= P(45, T) - P(50, T) - S + 50e^{-rT} + S
\]

\[
\leq 0
\]

\[
\leq 50e^{-rT}
\]

-5e^{-rT} \leq P(45, T) - P(50, T)

(III) is True