More on call/put prices as functions of the strike

$$V_c(K) \downarrow \quad \ldots$$

$$V_p(K) \uparrow \quad \ldots$$

with $$k_1 < k_2$$

$$V_c(k_1) - V_c(k_2) \leq PV(k_2-k_1) \quad \ldots$$

{ \begin{align*}
&\text{short the } k_1\text{-call} \\
&\text{long the } k_2\text{-call}
\end{align*} }

{ \text{Call Bear Spread} }

$$V_p(k_2) - V_p(k_1) \leq PV(k_2-k_1) \quad \ldots$$

{ \text{Put Bull Spread} }

{\text{arbitrage portfolio}}

Call Bull Spread

w/ $$k_1 < k_2$$

{ \begin{align*}
&\text{short the } k_1\text{-put} \\
&\text{long the } k_2\text{-put}
\end{align*} }

{ \text{Put Bear Spread} }

March 31st, 2015
12. You are given:

(i) \( C(K, T) \) denotes the current price of a \( K \)-strike \( T \)-year European call option on a nondividend-paying stock.

(ii) \( P(K, T) \) denotes the current price of a \( K \)-strike \( T \)-year European put option on the same stock.

(iii) \( S \) denotes the current price of the stock.

(iv) The continuously compounded risk-free interest rate is \( r \).

Which of the following is (are) correct?

\[ T \]

(I) \[ 0 \leq C(50, T) - C(55, T) \leq 5e^{-rT} \]

(II) \[ 50e^{-rT} \leq P(45, T) - C(50, T) + S \leq 55e^{-rT} \]

(III) \[ 45e^{-rT} \leq P(45, T) - C(50, T) + S \leq 50e^{-rT} \]

(A) (I) only

(B) (II) only

(C) (III) only

(D) (I) and (II) only

(E) (I) and (III) only

\[ V_p(45) - V_c(50) + S(0) = \]

\[ = V_p(45) - V_c(50) + V_p(50) - V_p(50) + S(0) = \]

\[ = V_p(45) - V_p(50) + (V_p(50) - V_c(50)) + S(0) = \]

\[ = V_p(45) - V_p(50) + (50e^{-rT} - F_{0,T}(S)) + S(0) = \]

\[ = V_p(45) - V_p(50) + 50e^{-rT} \leq 0 \]

\[ \Rightarrow (III) \] is true.
Problem #12.

Claims:
(I) \( 0 \leq V_C(50) - V_C(55) \leq 5e^{-rT} \)
(II) \( 50e^{-rT} \leq V_P(45) - V_C(50) + S(0) \leq 55e^{-rT} \)
(III) \( 45e^{-rT} \leq V_P(45) - V_C(50) + S(0) \leq 50e^{-rT} \)

Which is/are true?

I. TRUE?

\( V_P(45) - V_C(50) + S(0) = [\text{Put-call parity}] \)
\[ = V_P(45) - (V_P(50) + E_{0,T}(S) - 50e^{-rT}) \]
\[ + S(0) = S(0) = [\text{NO DIVIDENDS}] \]
\[ = V_P(45) - V_P(50) + 50e^{-rT} \]
\[ -5e^{-rT} \leq 0 \]
\[ V_P(50) - V_P(45) \leq 5e^{-rT} \]

\[ \Rightarrow 45e^{-rT} \leq V_P(45) - V_P(50) + 50e^{-rT} \leq 50e^{-rT} \]
\[ \Rightarrow \text{(III) is True.} \]
\[ \Rightarrow \boxed{\text{E.}} \]
**Convex Functions**

**Review:**

\[ y = f(x) \]

\[ f(x_1), f(x_2), f(x_3) \]

\[ x_1 < x_2 < x_3 \]

If \( x_1 < x_2 < x_3 \), then there exists \( 0 < \lambda < 1 \) such that:

\[ x_2 = \lambda x_1 + (1-\lambda) x_3 \]

\[ \iff \quad x_2 = \lambda x_1 + x_3 - \lambda x_3 \]

\[ \iff \quad x_2 - x_3 = \lambda (x_1 - x_3) \]

\[ \lambda = \frac{x_3 - x_2}{x_3 - x_1} \quad \text{and} \quad 1 - \lambda = \frac{x_2 - x_1}{x_3 - x_1} \]

**Def'n.** \( f \) is convex if

\[ f(\lambda x_1 + (1-\lambda) x_3) \leq \lambda f(x_1) + (1-\lambda) f(x_3) \]
Convexity of call/put prices w/ respect to the strike

\[ V_c(F_{0,T}(s)) \]

\[ \text{strike} \]

\[ K_1, K_2, K_3 \]

Claim:

\[(CC)\]

\[ V_c(K_2) \leq \lambda V_c(K_1) + (1 - \lambda) V_c(K_3) \]

w/ \[ \lambda = \frac{K_3 - K_2}{K_3 - K_1} \]

\[ \iff \]

\[ \frac{V_c(K_1) - V_c(K_2)}{K_2 - K_1} \geq \frac{V_c(K_2) - V_c(K_3)}{K_3 - K_2} \]

Assume to the contrary that there exist \[ K_1 < K_2 < K_3 \] such that

\[ V_c(K_2) > \lambda V_c(K_1) + (1 - \lambda) V_c(K_3) \]

I. \underline{Short it!}

II. Propose an ARBITRAGE PORTFOLIO

\[ \{ \]

- WRITE 1 \[ K_2 \]-strike call
- LONG \[ \lambda \] \[ K_1 \]-strike calls
- LONG \[ 1 - \lambda \] \[ K_3 \]-strike calls

\[ \} \]
Q: Would one use the call butterfly spread to hedge either a long, or a short inherent position w/ respect to the underlying? *No.

It lacks directionality, so no.

If not, then ASYMMETRIC.

Our portfolio is called a CALL BUTTERFLY SPREAD.

Call prices are convex as functions of the strike.

\[
\text{Profit} = \gamma \left( K_2 - K_3 \right)
\]

\[
\text{Payoff (Portfolio)} = 0
\]

\[
\text{Initial inflow of money:}
\]

\[
\text{Initial Cost: } N \times [c(K_2) + \left(1-N\right) V_c(K_3) - V_c(K_2)] < 0
\]
Put payoff: \((K - S(T))^+\)

If \(K = 0\) : \((0 - S(T))^+ = 0\).

\((PC)\) \(V_P(K_2) \leq \lambda V_P(K_1) + (1-\lambda) V_P(K_3)\) w/ \(\lambda = \frac{K_3 - K_2}{K_3 - K_1}\)

\[\iff\]

\(\frac{V_P(K_2) - V_P(K_1)}{K_2 - K_1} \leq \frac{V_P(K_3) - V_P(K_2)}{K_3 - K_2}\)

The ARBITRAGE PORTFOLIO in the case that \((PC)\) does not hold:

- Short 1 \(K_2\)-put
- Long \(\lambda\) \(K_1\)-puts
- Long \((1-\lambda)\) \(K_3\)-puts

Put BUTTERFLY SPREAD

Tasks for @ home:

1° Draw the payoff curve.

2° Compare it to the payoff curve of the call butterfly spread w/ the same \(K_1, K_2, K_3\).

3° Compare the initial costs between the call and the put butterfly spreads.
Speculating on Volatility

- Options can be used to create positions that are nondirectional with respect to the underlying asset.

- Examples
  - Straddles
  - Strangles
  - Butterfly spreads

- Who would use nondirectional positions?
  - Investors who do not care whether the stock goes up or down, but only how much it moves, i.e., who speculate on volatility.
STRADDLE

- a LONG call
- a LONG put

\( w/ \) identical inputs

The payoff of the straddle: \( Y(T) = |S(T) - K| \)

...A bet on HIGH VOLATILITY...

Initial cost: \( V_c(0) + V_p(0) \)
Strangles

- Buying an out-of-the-money call and put with the same time to expiration

- A strangle can be used to reduce the high premium cost, associated with a straddle
\[ K_p < K < K_c \]

\[ \{ \text{STRANGLE} \} \]

\[ \begin{align*}
K_p - \text{strike put} \\
+ \\
K_c - \text{strike call}
\end{align*} \]

\[ \{ \text{STRADDLE} \} \]

\[ \begin{align*}
K - \text{strike put} \\
+ \\
K - \text{strike call}
\end{align*} \]

\[ \begin{align*}
V_p (K_p) &\leq V_p (K) \\
+ \\
V_c (K_c) &\leq V_c (K)
\end{align*} \]

\[ \text{Cost of strangle} \leq \text{Cost of straddle}. \]