Futures Contracts

- Standardization ⇨ Futures contract more liquid ⇨ It's easier to buy/sell them. ⇨ We are confident of the "accuracy" of the observed futures price.
  ⇨ We can use futures contracts as an UNDERLYING ASSET for a call or a put.

- Credit risk alleviated by means of a MARGIN ACCOUNT.

**Notional Value** of the investment:

\[ N = n \times \text{Size of Index} \times \text{Futures Price} \]

\[ \uparrow \]

\[ \# \text{ of futures contracts} \]

**Margin Account(s)**... think of it as a savings acct, i.e.,

- **Earnings Interest** (@ the prevailing interest rate \( r \))

**Initial Margin**... the amount which needs to be initially deposited into the margin acct; usually expressed as a percentage of \( N \)
Settlement Dates

\[ B^s(t_{k+1}) \]

\[ = B^s(t_k^-) \cdot e^{r(t_k-t_k^-)} \]

Left-hand limit

\[ t_{k-1} \quad t_k \]

Right-hand limit

[The seller's perspective]

The effect of interest earned.

\[ B^s(t_k) = B^s(t_k^-) - \eta \cdot \text{Size of Index} (F_{t_k,T} - F_{t_{k-1},T}) \]

Change in the Futures price!

MARKING TO MARKET

The Maintenance Margin:

The comfort zone boundary of your broker; i.e., the minimal balance in the margin account (s)he is willing to accept.

\[ B^s(t_k^+) = \max(B^s(t_k), MM) \]

IF \( B^s(t_k) < MM \), THEN

a. MARGIN CALL IS MADE.
Options on Futures Contracts

Futures contracts w/ delivery date $T_F$.

Futures prices: $F_{t,T_F}$ for $0 \leq t \leq T_F$

are observable quantities.

Most Common: CALLS and PUTS

Derivative Securities.

Exercise date for the option

A Call on a futures contract:

- On the date $t$ at which the option is exercised:
  - the option’s holder acquires a LONG position in the futures contract;
  - receives the amount $\left( F_{t,T_F} - K \right)_+$

  the call option’s strike.
The maintenance margin: MM
\[ B^s(t_k^+) = \max (B^s(t_k), MM) \]

IF \( B^s(t_k) < MM \), then a margin call is made.

Options on Futures Contract

**Call**

On the date \( t \) on which the option is exercised, the option’s holder:
* acquires a **LONG** position in the underlying futures contract
* receives the amount \( (F_{t,T_F} - K)_+ \)

**Put**

* enters a **short** position in the underlying futures contract
* receive \( (K - F_{t,T_F})_+ \)
A Put: • a short position in the futures contract, • receive the amt: \((K - F_{t,T_F})_+\)

For a pair of otherwise identical futures put/call, we have the put-call parity:

\[
V_c(0) - V_p(0) = F_{0,T}^P (\text{underlying}) - Ke^{-rT}
\]

the futures contract for delivery at time-\(T_F\).

Futures price = Forward price

\[
V_c(0) - V_p(0) = e^{-rT} (F_{0,T_F} - K)
\]

Mnemonic device: \(S \leftrightarrow r\)

\(\text{dividend yield} \leftrightarrow \text{continuously compounded, risk-free interest rate}\)

Q: What if the options are at-the-money?

\[
V_c(0) = V_p(0)
\]
Binomial pricing

... on futures on stocks

**STOCK-PRICETREE** / Futures (forward)-price tree

\[ S(0) \]

\[ S_u = u \cdot S(0) \]

\[ S_d = d \cdot S(0) \]

\[ F_{0,T_F} \]

\[ S(0) \]

\[ S(t) = S(t) e^{(r-s)(T_F-t)} \]

The no-arbitrage forward price \((t \leq T_F)\): \( F_{t,T_F}(S) = S(t) e^{(r-s)(T_F-t)} \)

In particular:

\[ F_{0,T_F} = F_{0,T_F}(S) = S(0) e^{(r-s)T_F} \]

@ the up-node:

\[ S_u \cdot e^{(r-s)(T_F-t)} = u \cdot S(0) e^{(r-s)T_F} \cdot e^{-(r-s)h} = F_{0,T_F} \]

\[ = F_{0,T_F} \cdot u \cdot e^{-(r-s)h} \]

@ the down-node:

\[ S_d \cdot e^{(r-s)(T_F-t)} = F_{0,T_F} \cdot d \cdot e^{-(r-s)h} \]

\[ u_F \quad \text{... up factors for the futures-price tree} \]

\[ d_F \quad \text{... down} \]
The risk-neutral probability:

\[ p^* = \frac{e^{(r-g)h} - d}{u - d} = \frac{e^{(r-g)h} - d_F e^{(r-g)h}}{u_F e^{(r-g)h} - d_F e^{(r-g)h}} = \frac{1 - d_F}{u_F - d_F} \]
SAMPLE MFE

46. You are to price options on a futures contract. The movements of the futures price are modeled by a binomial tree. You are given:

(i) Each period is 6 months. \( \Delta = \frac{1}{2} \)
(ii) \( u/d = 4/3 \), where \( u \) is one plus the rate of gain on the futures price if it goes up, and \( d \) is one plus the rate of loss if it goes down.
(iii) The risk-neutral probability of an up move is 1/3.
(iv) The initial futures price is 80. \( F_{0,T_F} = 80 \)
(v) The continuously compounded risk-free interest rate is 5%. \( r = 0.05 \)

Let \( C_I \) be the price of a 1-year 85-strike European call option on the futures contract, and \( C_{II} \) be the price of an otherwise identical American call option.

Determine \( C_{II} - C_I \).

\[
\frac{1}{3} = p^* = \frac{1 - d_F}{u_F - d_F}
\]

\[
\frac{1}{3} = \frac{1 - d_F}{u_F - d_F}
\]

\[
\frac{1}{3} = \frac{1 - d_F}{u_F - d_F}
\]

(A) 0
(B) 0.022
(C) 0.044
(D) 0.066
(E) 0.088

1st The tree

2nd \( C_I = ? \), \( C_{II} = ? \) as usual

3rd \( C_{II} - C_I = \text{answer} \).
\[
\frac{1}{3} = p^* = \frac{1 - d_F}{u_F - d_F} \cdot \frac{d_F}{d_F} = \frac{1}{d_F} - 1
\]

\[
\frac{1}{d_F} - 1 = \frac{1}{q} \quad \Rightarrow \quad d_F = \frac{9}{10} \quad \Rightarrow \quad u_F = \frac{12}{10}
\]

The futures price tree:

\[
F_{0,T_F} = 80 \quad \leftarrow \quad 96 \quad \rightarrow \quad 115.20 \quad \leftarrow \quad 30.20
\]

\[
86.40 \quad \leftarrow \quad 72 \quad \rightarrow \quad \leftarrow \quad 85 \quad \rightarrow \quad 0
\]

Difference \( C_I^d - C_{II}^d = 0 \)

\( \left\langle k = 85 \quad \right\rangle \)

Price \( C_I \) and \( C_{II} \) \( \Rightarrow \) answer = \( C_{II} - C_I \)

Try to work efficiently (focus on the difference)!