The Binomial Asset-Pricing Model

Ambitious? Propose a stock-price model? 

Examples.

**Purpose:** PRICING OPTIONS on the STOCK.

\[ S(T) \sim N(\text{mean} = \mu, \text{var} = \sigma^2) \]... can be negative, so we would not use it outright.

\[ S(T) \sim U(50, 150) \]

\[ \text{pdf.} \]

\[ 50 \quad 150 \]

**DISCRETE MODEL**

A one-period binomial tree

\( T \): time horizon

\( S(T) \): stock price at time \( T \)

\( S(0) \): current (spot) asset price

\( S_u \) "up" node

\( S_d \) "down" node

The two possible values of \( S(T) \).
Usually: \[ Su = u \cdot S(0) \] \quad u \text{... up factor}
\[ Sd = d \cdot S(0) \] \quad d \text{... down factor}

Note: \[ u = \frac{Su}{S(0)} = \frac{Su-S(0)}{S(0)} + 1 \]
\[ d = \frac{Sd}{S(0)} = \frac{Sd-S(0)}{S(0)} + 1 \]

Assume \( r \)... continuously compounded, risk-free interest rate

**STOCK-PRICE TREE**

\[
\begin{array}{c}
S(0) \\
\downarrow \\
Su = u \cdot S(0) \text{ By convention } d < u \\
\text{risk-free investment} \\
Sd = d \cdot S(0) \\
\end{array}
\]

\[ h \]

If \( S(0) \) is invested \( @ \) the risk-free rate \( r \),

**THEN** the balance \( @ \) time \( h \) is \( S(0)e^{rh} \).

We only work \( w/ \) continuous-dividend-paying stocks;
set \( \delta \)... dividend yield.

\# of \( \) shares owned \( @ \) time \( h \): \( e^{\delta h} \)

IF "up" : \( e^{\delta h} \cdot u \cdot S(0) \)

IF "down" : \( e^{\delta h} \cdot d \cdot S(0) \)

We suspect that the no-arbitrage condition is

\[ e^{\delta h} d \leq e^{rh} \leq e^{\delta h} u \]

\[ \iff \quad d \leq e^{(r-\delta)h} \leq u \]

The NO-ARBITRAGE condition for the binomial asset-pricing model.
Assume TO THE CONTRARY that
\[ e^{(r-d)h} \leq d < u \]

I. Suspicion.

II. Construction of an arbitrage portfolio:
- Buy 1 share of stock.

III. Verification:
- Initial cost: \( S(0) \)
- Payoff: \( e^{\delta h} \cdot S(T) \)

IF "up": \( e^{\delta h} \cdot u \cdot S(0) - e^{\delta h} S(0) = (e^{\delta h} u - e^{\delta h}) S(0) > 0 \)

IF "down": \( e^{\delta h} \cdot d \cdot S(0) - e^{\delta h} S(0) = (e^{\delta h} d - e^{\delta h}) S(0) > 0 \)

ARBITRAGE PORTFOLIO.
The payoff of a derivative security is

\[ V(T) = \psi(S(T)) = \psi(S(h)) \]

\[ \text{PAYOFF FUNCTION} \]

\[ \text{one period} \]

\[ \Rightarrow T = h \]

**Example.** Let our derivative security be a call w/ strike \( K \)

\[ \Rightarrow \psi(s) = (s-K)^+ \]

and \[ V(T) = (S(T) - K)^+ \]

\[ \text{DERIVATIVE SECURITY TREE} \]

\[ \text{STOCK-PRICE TREE} \]

Knowing the dist’n of \( S(T) \) gives us full info about the dist’n of \( V(T) \).