The Binomial Asset-Pricing Model

Ambitious? Propose a stock-price model?

What's the purpose of this model?

- What's the investment decision (subjective)?
  - e.g., stock prices should be modeled as random

Generally, what are the desirable attributes of a model?

- have the crucial properties of the modeled phenomenon
- useful/transparent/simple

Depends on the stock price behaviour.

Consider: Pricing derivative securities.

\[ \begin{align*}
    S(0) & \quad \text{observable} \\
    \text{current/spot asset price} & \\
    \text{"root" node} & \\
    S_d & \quad \text{"down" node} \\
    \text{"up" node} & \quad \text{offspring} \\
    \text{Su} & \\
    \text{Su} = u \cdot S(0) & \\
    \text{Sd} & = d \cdot S(0) & \text{u... up factor} & \text{d... down factor} \\
    T \ldots \text{expiration data} & \\
    S(T) & \quad \text{rand var denoting the stock price @ time-T} \\
    T & \end{align*} \]
\[
\begin{align*}
\Rightarrow \quad u &= \frac{S_{u}}{S(o)} = \frac{S_{u} - S(o)}{S(o)} + 1 \\
d &= \frac{S_{d}}{S(o)} = \frac{S_{d} - S(o)}{S(o)} + 1
\end{align*}
\]

Assume \( r \) ... continuously compounded, risk-free interest rate.

\[
S(o) \xrightarrow{\text{risk-free investment}} S_{u} = u \cdot S(o)
\]

\[
S(o) \xrightarrow{\text{risk-free investment}} S_{d} = d \cdot S(o)
\]

\[
T = T
\]

If \( S(o) \) is invested \( @ \) the risk-free rate \( r \),

then the balance \( @ \) time- \( h \) is \( S(o) e^{rT} \).

If \( S(o) \) is invested in the stock, then?

We only work w/ continuous-dividend-paying stocks; \( \delta \ldots \) dividend yield.

\# of shares owned at time- \( h \): \( e^{\delta h} \)

The wealth is based on the movement of the stock price:

IF "up" \( e^{\delta h} \cdot u \cdot S(o) \)

IF "down" \( e^{\delta h} \cdot d \cdot S(o) \)
We suspect the no-arbitrage condition to be:

\[ e^{\delta h} d S(0) < e^{r h} S(0) < e^{\delta h} u S(0) \]

\[ \iff \quad d < e^{(r-\delta)h} < u \]  

(NA)

Assume to the contrary that

\[ e^{(r-\delta)h} \leq d < u. \]

I. Suspicion.  

II. Construction of an arbitrage portfolio:

\[ \text{BUY} \quad 1 \text{ SHARE OF STOCK} \]

III. Verification:

- Initial cost: \( S(0) \)
- Payoff: \( e^{\delta h} S(h) \)

**Profit:**

IF "up": \( e^{\delta h} u S(0) - e^{r h} S(0) = (e^{\delta h} u - e^{r h}) S(0) \)

\[ = e^{\delta h} (u - e^{(r-\delta)h}) S(0) > 0 \]

IF "down": \( e^{\delta h} d S(0) - e^{r h} S(0) = (e^{\delta h} d - e^{r h}) S(0) > 0 \leftarrow \text{ARBITRAGE} \)