THE BINOMIAL OPTION-PRICING PROCEDURE

Focus on a particular derivative security w/ the payoff function v.

THE STOCK-PRICE TREE

$$S(0) \begin{cases} S_u \\ S_d \end{cases}$$

Populated FORWARD in time.

THE DERIVATIVE-SECURITY TREE

$$\boxed{V(0)} \begin{cases} V_u = v(S_u) \\ \boxed{V_d = v(S_d)} \end{cases}$$

?

Populated BACKWARD in time.

our Market-model permits a REPLICATING PORTFOLIO consisting of:
- a investment in the risky asset S
- a risk-free investment

FOR ALL DERIVATIVE SECURITIES.

More precisely, the replicating portfolio will consist of:
- Δ units of the underlying
- B invested at the risk-free rate r

δ... dividend yield.

THE REPLICATING-PORTFOLIO TREE

$$\underbrace{\Delta \cdot S(0) + B \begin{cases} \Delta e^{\delta h} \cdot S_u + Be^{rh} \\ \boxed{\Delta e^{\delta h} \cdot S_d + Be^{rh}} \end{cases}}_{h}$$

Within our model, the portfolio is, indeed, replicating $\mathcal{P}$:

$$\boxed{\begin{aligned} V_u &= \Delta e^{\delta h} \cdot S_u + B e^{rh} \\ V_d &= \Delta e^{\delta h} \cdot S_d + B e^{rh} \end{aligned}} \Longrightarrow V(0) = \Delta S(0) + B$$

NO ARBITRAGE!

Solve for Δ and B.

$$V_u - V_d = \Delta e^{\delta h} \cdot S_u - \Delta e^{\delta h} S_d = \Delta e^{\delta \cdot h}(S_u - S_d)$$

$$\Rightarrow \Delta = \underbrace{e^{-\delta h}} \cdot \frac{V_u - V_d}{S_u - S_d}$$

$$\Rightarrow B e^{rh} = V_u - \Delta e^{\delta h} S_u =$$

$$= V_u - \frac{V_u - V_d}{S_u - S_d} \cdot S_u = V_u - \frac{V_u - V_d}{\cancel{S(0)}(u-d)} \cdot \cancel{S(0)} \cdot u$$

$$= \frac{\cancel{uV_u} - dV_u - \cancel{uV_u} + uV_d}{u-d} = \frac{uV_d - dV_u}{u-d}$$

$$\Rightarrow \boxed{B = e^{-rh} \cdot \frac{uV_d - dV_u}{u-d}}$$

Pricing:

$$V(0) = \Delta \cdot S(0) + B$$

$$= e^{-\delta h} \cdot \frac{V_u - V_d}{\underbrace{S_u - S_d}_{\cancel{S(0)}(u-d)}} \cdot \cancel{S(0)} + e^{-rh} \cdot \frac{uV_d - dV_u}{u-d}$$

$$= \frac{1}{u-d}\left[e^{-\delta h}(V_u - V_d) + e^{-rh}(uV_d - dV_u)\right]$$

$$= e^{-rh} \cdot \frac{1}{u-d}\left[e^{(r-\delta)h}(V_u - V_d) + uV_d - dV_u\right]$$

$$= e^{-rh} \cdot \frac{1}{u-d}\left[V_u\left(e^{(r-\delta)h} - d\right) + V_d\left(u - e^{(r-\delta)h}\right)\right]$$

$$= e^{-rh} \cdot \left[V_u \cdot \frac{e^{(r-\delta)h} - d}{u-d} + V_d \cdot \frac{u - e^{(r-\delta)h}}{u-d}\right]$$

ADD UP TO 1 !!!

BOTH POSITIVE !!!

RISK-NEUTRAL PROBABILITY MEASURE

$$p^* = \frac{e^{(r-\delta)h} - d}{u-d}$$

... the risk-neutral probability of the stock price going up.

$$V(0) = e^{-rT}\left[p^* \cdot V_u + (1-p^*)V_d\right] = e^{-rT}\,\mathbb{E}^*\left[V(T)\right]$$

COME TOGETHER!

<u>Example</u>. Derivative security = the stock itself.

Assume no dividends.

$h = T, \delta = 0$

$$V(0) = e^{-rT}\left[p^* \cdot S_u + (1-p^*) S_d\right]$$

$$= e^{-rT}\left[\frac{e^{(r-\delta)h} - d}{u-d} \cdot u \cdot S(0) + \frac{u - e^{(r-\delta)h}}{u-d} \cdot d \cdot S(0)\right]$$

$$= S(0)$$

14. For a one-year straddle on a nondividend-paying stock, you are given:

(i) The straddle can only be exercised at the end of one year.

(ii) The payoff of the straddle is the absolute value of the difference between the strike price and the stock price at expiration date. $|S(T) - K|$

(iii) The stock currently sells for \$60.00.

(iv) The continuously compounded risk-free interest rate is 8%.

(v) In one year, the stock will either sell for \$70.00 or \$45.00. $S(0) = 60$, $S_u = 70$, $S_d = 45$

(vi) The option has a strike price of \$50.00.

Calculate the current price of the straddle.

(A) \$0.90

(B) \$4.80

(C) \$9.30

(D) \$14.80

(E) \$15.70

Steps: 1st The payoff at time-T, i.e., fill up the "leaves" of the option tree!

$V(0)$? $V_u = |70 - 50| = 20$, $V_d = |45 - 50| = 5$

2nd THE Risk-neutral Probability

$$p^* = \frac{e^{(r-\delta)h} - d}{u - d}.$$

$$p^* = \frac{e^{rh} - \frac{S_d}{S(0)}}{\frac{S_u}{S(0)} - \frac{S_d}{S(0)}} = \frac{S(0)e^{rh} - S_d}{S_u - S_d}$$

3rd Employ the risk-neutral pricing formula.