Graphical Interpretation of the Replicating Portfolio

**Special case:** A CALL OPTION on a NON-DIVIDEND-PAYING asset.

\[
\text{PAYOFF}\quad \frac{V_u}{V_d} = \frac{(S_u-K)_+}{(S_d-K)_+}
\]

The cord goes through \((S_d,V_d)\) and \((S_u,V_u)\)

- slope: \(\frac{V_u-V_d}{S_u-S_d} = \Delta > 0 \Rightarrow \text{Buy shares.}\)
- The intercept: \(B \times e^{rT} < 0 \Rightarrow \text{Borrowing @ risk-free rate } r\).

**The Risk-Neutral Pricing Formula**

\[
V_c(0) = e^{-rT} \left[ p^*(S_u-K)_+ + (1-p^*)(S_d-K)_+ \right]
\]

\[
V_p(0) = e^{-rT} \left[ p^*(K-S_u)_+ + (1-p^*)(K-S_d)_+ \right]
\]

\[\text{with } p^* = \frac{e^{(r-\delta)h} - d}{u-d}\]

Q: What is \(V_c(0) - V_p(0)\)?
3. You are given the following regarding stock of Widget World Wide (WWW):

(i) The stock is currently selling for $50.

(ii) One year from now the stock will sell for either $40 or $55.

(iii) The stock pays dividends continuously at a rate proportional to its price. The dividend yield is 10%.

\[ \delta = 0.10 \]

The continuously compounded risk-free interest rate is 5%.

While reading the Financial Post, Michael notices that a one-year at-the-money European call written on stock WWW is selling for $1.90. Michael wonders whether this call is fairly priced. He uses the binomial option pricing model to determine if an arbitrage opportunity exists.

\[ T = 1, K = 50, V^{\text{obs}}_c(o) = 1.90 \]

What transactions should Michael enter into to exploit the arbitrage opportunity (if one exists)?

(A) No arbitrage opportunity exists.

(B) Short shares of WWW, lend at the risk-free rate, and buy the call priced at $1.90.

(C) Buy shares of WWW, borrow at the risk-free rate, and buy the call priced at $1.90.

(D) Buy shares of WWW, borrow at the risk-free rate, and short the call priced at $1.90.

(E) Short shares of WWW, borrow at the risk-free rate, and short the call priced at $1.90.

\[ V_c(o) = 2.4 > 1.9 = V^{\text{obs}}_c(o) \implies \text{\(\Box\)} \]
\[ V_r(0) \overset{?}{=} V_{obs}(0) \]

- Yes: arbitrage?
  - No arbitrage?
    - \[ V(0) > V_{obs}(0) \]
      - Buy the observed one
      - Short the replicating portfolio
  - Yes arbitrage?
    - \[ V(0) < V_{obs}(0) \]
      - Write the observed
      - Long the replicating portfolio
The Forward Binomial Tree

\( u, d = \) ?

\( \sigma \ldots \) the standard deviation of the continuously compounded rate of return on the annual time-scale

Toy example: \( h \ldots \) one month, i.e., \( \frac{1}{12} \)

\( \sigma_h \ldots \) monthly volatility

\[
\begin{align*}
0 & \quad \frac{1}{12} & \quad \frac{2}{12} & \quad \ldots & \quad \frac{11}{12} & \quad 1 \\
\end{align*}
\]

\( \text{REALIZED RETURNS} \)

Assume:

- Rates of return are INDEPENDENT and IDENTICALLY DISTRIBUTED over DISJOINT time period of EQUAL LENGTH.

\[
\sigma^2 = 12 \cdot \sigma_h^2 \quad \Rightarrow \quad \sigma_h = \sigma \sqrt{\frac{1}{12}}
\]

\( \sigma_h = \sigma \sqrt{h} \)

Recall:

\[
F_{0,h}(S) = S(0) e^{(r-\sigma)h}
\]

In the forward tree:

\[
\begin{align*}
S_u &= F_{0,h}(S) e^{\sigma \sqrt{h}} = S(0) e^{(r-\sigma)h + \sigma \sqrt{h}} \\
S_d &= F_{0,h}(S) e^{-\sigma \sqrt{h}} = S(0) e^{(r-\sigma)h - \sigma \sqrt{h}}
\end{align*}
\]

\( =: u \)

\( =: d \)
• $\frac{S_u}{S_d} = e^{2\sigma \sqrt{t}}$

• Risk-neutral probability $p^* = ?$

• Q: Is there a need to check that the No-Arbitrage condition holds?