Asian Options [cont'd]

\[ (S(t_1), \ldots, S(t_k), \ldots, S(t_n)) \longrightarrow \left\{ \begin{array}{l}
A(T) := \frac{1}{n} \sum_{k=1}^{n} S(t_k) \\
G(T) = \left( \prod_{k=1}^{n} S(t_k) \right)^{1/n}
\end{array} \right. \]

\[ A(T) \geq G(T) \]

<table>
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<tr>
<th>PAYOFFS</th>
<th>GEOMETRIC</th>
<th>ARITHMETIC</th>
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| STRIKE  | \begin{align*}
\text{Call:} & \quad (S(T) - G(T))_+ \\
\text{Put:} & \quad (G(T) - S(T))_+
\end{align*} & \begin{align*}
\text{Call:} & \quad (S(T) - A(T))_+ \\
\text{Put:} & \quad (A(T) - S(T))_+
\end{align*} |
| UNDERLYING ASSET & \begin{align*}
\text{Call:} & \quad (G(T) - K)_+ \\
\text{Put:} & \quad (K - G(T))_+
\end{align*} & \begin{align*}
\text{Call:} & \quad (A(T) - K)_+ \\
\text{Put:} & \quad (K - A(T))_+
\end{align*} |

Draw analogous inequalities w/ option prices.
Problem. Non-dividend-paying stock has $S(0) = 100$.

\[ \sigma = 0.3 \]
\[ r = 0.04 \]

We model the stock price over the next year with a two-period forward binomial tree.

\[ u = e^{(r - \delta)h + \sigma \sqrt{h}} \]
\[ d = e^{(r - \delta)h - \sigma \sqrt{h}} \]

w/ $h = \frac{1}{2}$ and $S = 0$.

\[ u = e^{0.04 \cdot 0.5 + 0.3 \sqrt{0.5}} \approx 1.26 \]
\[ d = e^{0.02 - 0.3 \sqrt{0.5}} \approx 0.83 \]

\[ S_0 = 100 \]

\[ Su = 126 \]
\[ Sd = 83 \]

\[ S_{dd} = 68.89 \]

\[ p^* = \frac{1}{1 + e^{\sigma \sqrt{h}}} = \frac{1}{1 + e^{0.3 \sqrt{0.5}}} \approx 0.447 \]

Consider an Asian geometric-average-strike call w/ exercise date in one year. What is its price consistent w/ the above stock-price tree?

**PAYOFF:** $(S(T) - G(T))^+$
\[ V_{uu} = (S_{uu} - G_{uu})_+ = (S(0) \cdot u^2 - \sqrt{S(0) \cdot u \cdot S(0)} \cdot u^2)_+ = S(0) \cdot u (u - \sqrt{u})_+ = 17.33 \]

\[ V_{ud} = (S_{ud} - G_{ud})_+ = S(0) \cdot u (d - \sqrt{d'})_+ = 0 \]

\[ V_{du} = (S_{du} - G_{du})_+ = (S(0) \cdot u \cdot d - \sqrt{S(0) \cdot d \cdot S(0) \cdot u \cdot d'})_+ = S(0) \cdot d \cdot (u - \sqrt{u})_+ = 11.41 \]

\[ V_{dd} = 0 \]

\[ V(0) = e^{-0.04} \left[ (0.447)^2 \cdot 17.33 + 0.447 \cdot (1 - 0.447) \cdot 11.41 \right] \]

\[ V(0) = 6.04 \]
Barrier Options

If the stock price touches/crosses the barrier $H$ before or at the exercise date, then the barrier option changes its state.

"Dead" $\rightarrow$ "Alive"  "Alive" $\rightarrow$ "Dead"

\[ \text{KNOCK IN} \quad \text{KNOCK OUT} \]

THE PAYOFF is paid on the exercise date $T$.

Example: REBATE OPTIONS

A fixed amount $R$ is paid to the option’s owner at time $T$ if the barrier $H$ is ever touched/crossed by the stock price.

\[
\begin{align*}
M(T) &= \max_{0 \leq t \leq T} S(t) \\
m(T) &= \min_{0 \leq t \leq T} S(t)
\end{align*}
\]