**NOV 4th**

- 10 Min Quiz today
- bonus pts due on Fri
- HW due on Monday (11/09)
- NOV 11th: .til "box spreads" (inclusive)
  - Sample exam is posted
16.1. **Introduction.** We wish to introduce the simplest model for the price of a risky asset at a specific time in the future. Were we to assume that price to be deterministic, the asset itself would no longer be risky. The least number of possible values the asset price can take at time $-h$ is, of course, two. We can depict this model using a simple tree as follows:

![Diagram of a binomial asset-pricing model]

The vertices in the above tree are called the **nodes**. The left-most node is frequently referred to as the **root node**. Its label $S(0)$ will correspond to the current (spot) asset price observed at time $-0$. Our modeling assumption that there are two possible asset prices at time $-h$ translate into the root node having two **offspring**: the “up” node and the “down” node. The asset price at time $-h$, denoted by $S(h)$ is, hence, a random variable with two possible values $S_u$ and $S_d$. It is customary for these values to be arranged so that $S_u > S_d$. Thus, we end up with the **one-period binomial asset-price tree**. The time length $h$ is called the length of a single period (or step) of the binomial tree.

Frequently, the model above is posited not by stipulating the values $S_u$ and $S_d$, but via a pair of constants $u$ and $d$. The “up” factor $u$ and the “down” factor $d$ satisfy the following
equalities:

\[ S_u = uS(0) \quad \text{and} \quad S_d = dS(0). \]

It is evident that the constants \( u \) and \( d \) are related to the possible values of the realized simple rate of return of the risky asset \( S \). Indeed, the realized simple rate of return over the period \([0, h]\) is given by

\[ \frac{S(h) - S(0)}{S(0)}. \]

Within the above model, the possible values that the realized simple rate of return can attain are:

\[ \frac{S_u - S(0)}{S(0)} \quad \text{and} \quad \frac{S_d - S(0)}{S(0)}. \]

The constants \( u \) and \( d \) can be expressed as

\[ u = \frac{S_u}{S(0)} = \frac{S_u - S(0)}{S(0)} + 1 \]

\[ d = \frac{S_d}{S(0)} = \frac{S_d - S(0)}{S(0)} + 1 \]

16.2. The no-arbitrage condition. The model above is simplicity itself and it is, at first sight, difficult to imagine that the constants \( u \) and \( d \) could possibly be chosen poorly. However, this is an issue worth looking into. First, we need to determine what it would mean for the pair \( u \) and \( d \) to be chosen poorly. Assume that the risky asset of interest is a continuous-dividend-paying stock. The existing environment at time\(-0\) when we build the model consists of the following:

- the continuously compounded risk-free interest rate \( r \),
- the initial asset price \( S(0) \),
- the dividend yield \( \delta \).

The binomial asset-pricing model will be ill-posed if \( u \) and \( d \) are chosen so that they introduce arbitrage in the already existing environment. The existence of an arbitrage opportunity hinges on the relationship between the risk-less and risky investment.

The risk-less investment. If the amount \( S(0) \) is invested at the continuously compounded risk-free interest rate \( r \), at the end of the investment period, the wealth of the investor will simply be equal to \( S(0)e^{rh} \).

The risky investment. If the amount \( S(0) \) is invested in the risky asset, the investor gets to purchase exactly one share of stock at time\(-0\). Due to continuous immediate reinvestment of the dividend in the same stock, the number of shares of stock owned at time\(-h\) equals \( e^{\delta h} \). The investor’s wealth is, hence, equal to \( S(h)e^{\delta h} \) – a random variable. The possible values of this random variable, i.e., the possible values the wealth of the shareholder can attain are:

- up: \( e^{\delta h}S_u = S(0)ue^{\delta h} \) if the stock price moved “up”, and
- down: \( e^{\delta h}S_d = S(0)de^{\delta h} \) if the stock price moved “down”.

It is logical to surmise that the investor who exposes himself to risk does so because (s)he is in a sense monetarily compensated for doing so. Plainly said, the wealth resulting from the
Market model

- **risk-less**: @ the continuously-compounded, risk-free \( r \)

- **risky asset**: continuous-dividend-paying stocks \( \delta \)

\[ S(0) \]

\[ Su = u \cdot S(0) \]
\[ Sd = d \cdot S(0) \]

\[ S(0)e^{rh} \]

\[ e^{Sh} \cdot Su = e^{Sh} \cdot u \cdot S(0) \] risk-free investment

\[ e^{Sh} \cdot Sd = e^{Sh} \cdot d \cdot S(0) \]

**WEALTH (1 Long share)**

\[ \Rightarrow \text{No-arbitrage condition} \]

\[ e^{Sh} \cdot d \cdot S(0) < e^{rh} \cdot S(0) < e^{Sh} \cdot u \cdot S(0) \]

\[ d < e^{(r-\delta)h} < u \]
Assume, to the contrary,

\[ e^{(r-g)h} \leq d < u \]

I. \( \checkmark \)

II. Propose the arbitrage portfolio:

- buy 1 share of stock

III. Verification

Initial Cost : \( S(0) \)

Payoff : \( e^{Sh} \cdot S(h) \)

\( \Rightarrow \) PROFIT : \( e^{Sh} \cdot S(h) - e^{r_h} \cdot S(0) \)

Two possible states of the world:

"up" node : \( e^{Sh} \cdot u \cdot S(0) - e^{r_h} \cdot S(0) = S(0)e^{Sh} (u-e^{(r-g)h}) \)

"down" node : \( e^{Sh} \cdot d \cdot S(0) - e^{r_h} \cdot S(0) = S(0)e^{Sh} (d-e^{(r-g)h}) \)

\( \geq 0 \)
Intro to Valuation

$S(0)$

$V(0) = ?$

$S_u = u \cdot S(0)$  $v(S_u) = V_u$

$S_d = d \cdot S(0)$  $v(S_d) = V_d$

$\uparrow$

The two possible values of the payoff

$S(0) = 100; \quad u = 1.04, \quad d = 0.96$

$S_u = 104$

$S_d = 96$

European, 100-strike call

$v(s) = (s - 100)^+$

$V_u = 4$

$V_d = 0$