Nov 6th AM

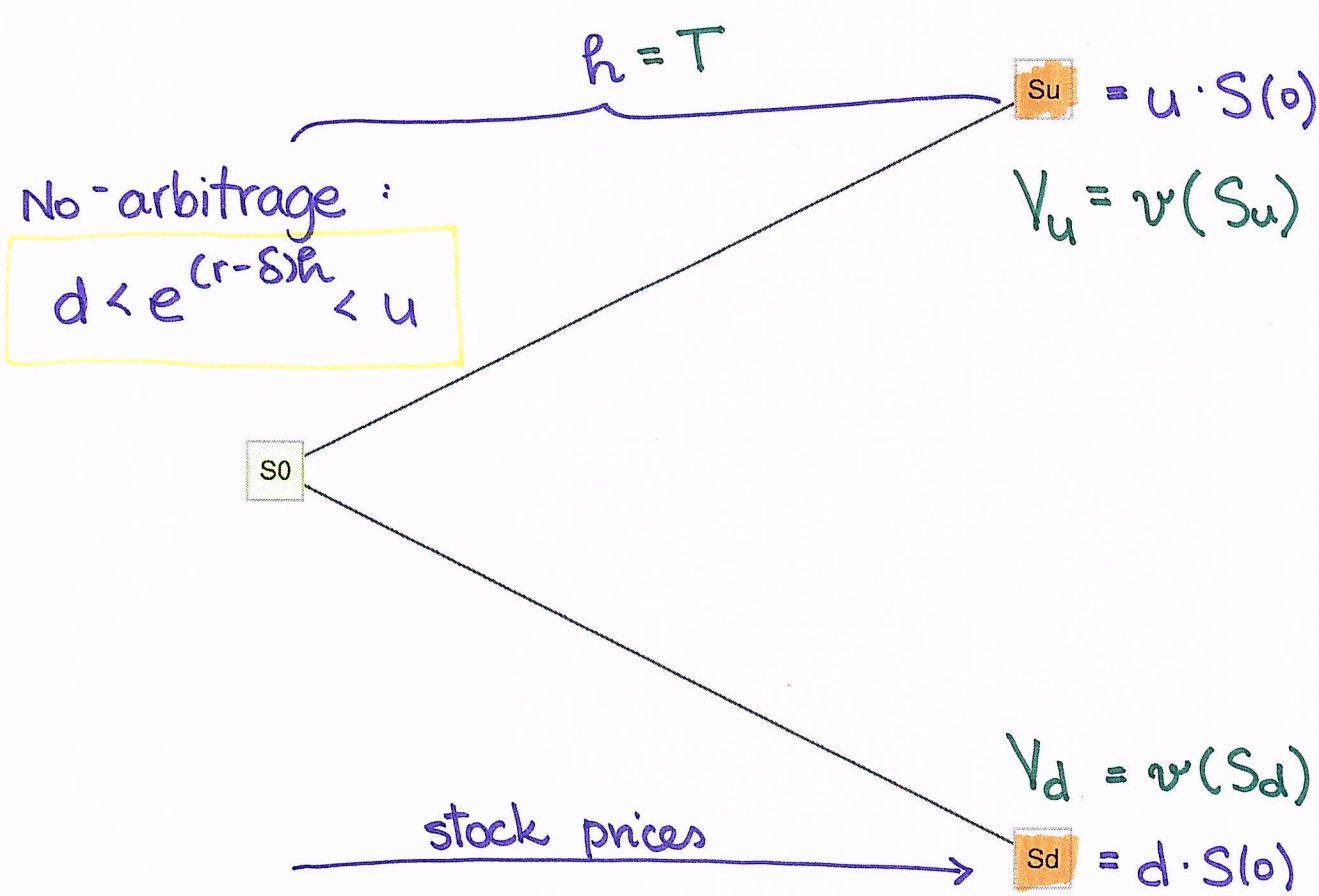
UNIVERSITY OF TEXAS AT AUSTIN

Lecture 17

Option pricing in the one-period binomial model.

17.1. **Introduction.** Recall the one-period binomial tree which we used to depict the simplest non-deterministic model for the price of an underlying asset at a future time$-h$.

$\boxed{u>d}$

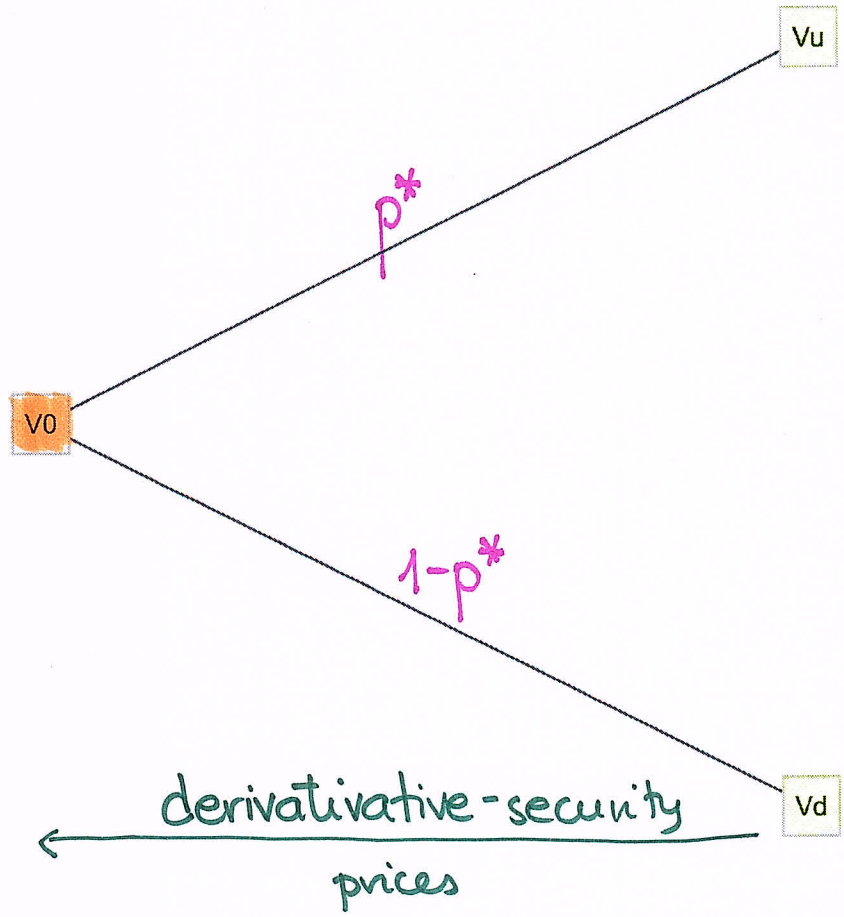


Our next objective is to determine the no-arbitrage price of a European-style derivative security with the exercise date T coinciding with the length h of our single period.

Consider such a derivative security whose payoff function is denoted by v. The payoff of this derivative security is, thus, a random variable

$$\boxed{V(T) = v(S(T)) = v(S(h)).}$$

Per our stock-price model above, the random variable $S(T)$ can only attain values S_u and S_d. So, the random variable $V(T)$ can only take the values $V_u := v(S_u)$ and $V_d := v(S_d)$. We can depict the resulting derivative-security tree as follows:



Note that we constructed the stock-price tree by starting from the root node containing the initial observed stock price. Then, we used our model encapsulated in the pair (u, d) of multiplicative factors to "populate" the offspring nodes. In short, we moved **from left to right**. Now that we wish to figure out the option price dictated by our stock-price tree, we start from the only known quantities: the possible payoffs. Then, we move *from right to left* to calculate the price of the derivative security occupying the root node of the derivative-security tree.

17.2. **Pricing by replication.** The method by which we intend to accomplish the above goal is the following:

B { B>0 ... lend/buy a bond; B<0 ... borrow

Step 1. Create the replicating portfolio for our derivative security consisting of an investment in the underlying risky asset and a loan (given or taken) at the continuously compounded risk-free interest rate r. ...

Δ shares { Δ>0 buy; Δ=0 no trading; Δ<0 short sell } in the stock

Step 2. Calculate the initial cost of the replicating portoflio.

Step 3. Conclude that the no-arbitrage price $V(0)$ of our derivative security must equal the initial cost of its replicating portfolio.

Let us, again, focus on the underlying asset being a continuous-dividend-paying stock with the dividend yield δ. It is traditional to denote by Δ the initial number of units of

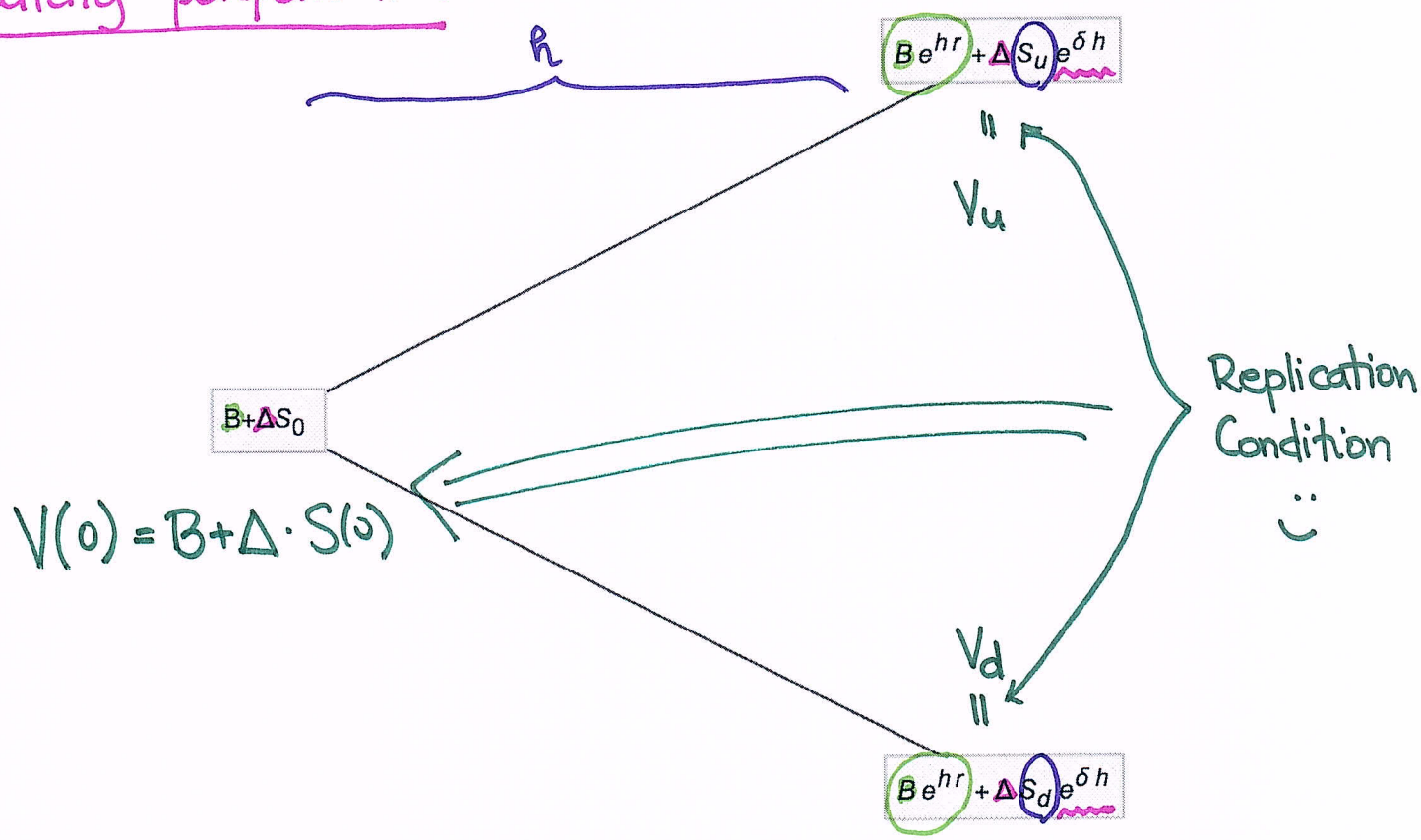
the underlying in the replicating portoflio.[1] If $\Delta > 0$, then Δ units of the underlying asset are longed, i.e., purchased. If $\Delta < 0$, then $|\Delta|$ units of the underlying asset are shorted. In particular, in case that the underlying asset is a stock, the negative value of Δ implies a short-sale of $|\Delta|$ shares of stock. We denote the initial amount invested at the continuously compounded risk-free interest rate r by B. The choice of notation here is obvious: B stands for bonds, seeing as zero-coupon bonds are a simple device for both lending and borrowing money. If $B > 0$, then the amount B is invested at the rate r. If $B < 0$, then the amount $|B|$ is borrowed at the continuously compounded risk-free interest rate r.

Recall that in the one-period model $T = h$. With the above notation and the convention that dividends are to be continuously and immediately reinvested in the same stock, we see that the number of shares "owned" at the end of the time interval $[0, T]$ equals $\Delta e^{\delta h}$. Likewise, the riskless investment accumulated to Be^{rh}. Hence, the total value of the replicating portfolio at time$-T$ is a random variable equal to

$$\Delta e^{\delta h} S(h) + Be^{rh}.$$

We can depict the two possible values that the replicating portfolio can attain using the following one-period binomial tree:

Replicating-portfolio tree



In order for the above portfolio to indeed be a **replicating portfolio** of the derivative security with the payoff function v, its payoff needs to be equal to the random variable $V(T)$

[1]The significance of the notation will be discussed in M339W.

$$\left.\begin{aligned} \Delta e^{\delta h} \cdot S_u + B e^{rh} &= V_u \\ \Delta e^{\delta h} \cdot S_d + B e^{rh} &= V_d \end{aligned}\right\} -$$

$$\Delta e^{\delta h}(S_u - S_d) = V_u - V_d$$

$$\boxed{\Delta = e^{-\delta h} \cdot \frac{V_u - V_d}{S_u - S_d}}$$

$$\cancel{e^{-\delta h}} \cdot \frac{V_u - V_d}{S_u - S_d} \cdot \cancel{e^{\delta h}} \cdot S_u + B e^{rh} = V_u$$

$$B e^{rh} = V_u - \frac{V_u - V_d}{\cancel{S(0)}(u-d)} \cdot \cancel{S(0)} \cdot u$$

$$\boxed{B = e^{-rh} \cdot \frac{u V_d - d \cdot V_u}{u - d}}$$

Pricing by replication:

$$V(0) = \Delta \cdot S(0) + B$$

$$V(0) = e^{-\delta h} \cdot \frac{V_u - V_d}{\frac{\cancel{S_u - S_d}}{u-d}} \cdot \cancel{S(0)} + e^{-rh} \cdot \frac{u V_d - d V_u}{u-d}$$

$$V(0) = e^{-rh} \cdot \frac{1}{u-d} \left[e^{(r-\delta)h} \cdot V_u - e^{(r-\delta)h} \cdot \boxed{V_d} + u \boxed{V_d} - d V_u \right]$$

$$V(0) = e^{-rh} \cdot \frac{1}{u-d} \left[V_u \left(e^{(r-\delta)h} - d \right) + V_d \left(u - e^{(r-\delta)h} \right) \right]$$

$$V(0) = e^{-rh} \left[V_u \cdot \frac{e^{(r-\delta)h} - d}{u-d} + V_d \cdot \frac{u - e^{(r-\delta)h}}{u-d} \right]$$

- both positive
- sum up to one

$$p^* := \frac{e^{(r-\delta)h} - d}{u-d}$$

THE RISK-NEUTRAL PROBABILITY of the stock price going up in a single period

$$\boxed{V(0) = e^{-rh} \underbrace{\left[V_u \cdot p^* + V_d (1-p^*) \right]}_{\mathbb{E}^*[V(T)]}}$$

Risk-neutral pricing formula

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Problem set #3

One-period binomial option pricing.

Problem 3.1. MFE Exam, Spring 2007: Problem #14

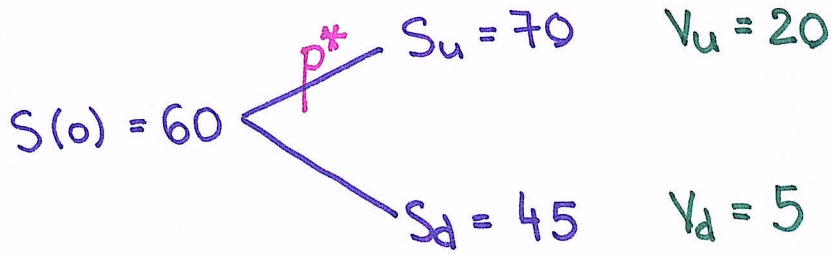
For a one-year straddle on a non-dividend-paying stock, you are given:

- The straddle can only be exercised at the end of one year. $T = 1$
- The payoff of the straddle is the absolute value of the difference between the strike price and the stock price at expiration date. $V_{STR}(T) = |S(T) - K|$, $K = 50$
- The stock currently sells for \$60.00. $S(0)$
- The continuously compounded risk-free interest rate is 8%. r
- In one year, the stock will either sell for \$70.00 or \$45.00. the one-period bin. tree
- The option has a strike price of \$50.00.

Calculate the current price of the straddle. $V_{STR}(0) = ?$

(A) \$0.90
(B) \$4.80
(C) \$9.30
(D) \$14.80
(E) \$15.70

PAYOFF



$$p^* = \frac{e^{(r-\delta)h} - d}{u-d} = ?$$

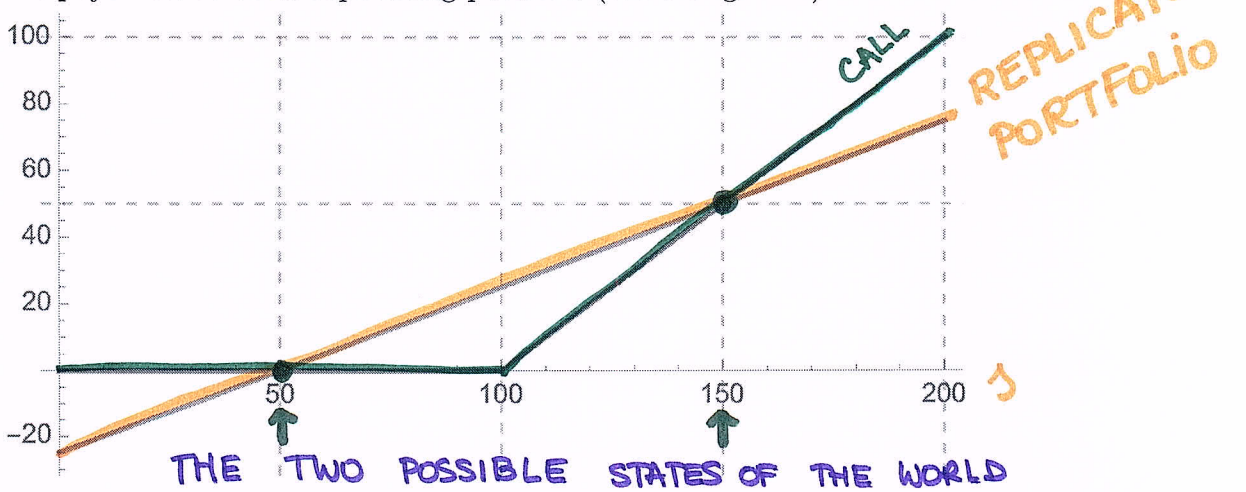
$$p^* = \frac{S(0)e^{rh} - S_d}{S_u - S_d} = \frac{60e^{0.08} - 45}{70 - 45} \approx 0.8 = \frac{4}{5}$$

$$V_{STR}(0) = e^{-0.08}\left[20 \cdot \frac{4}{5} + 5 \cdot \frac{1}{5}\right] = e^{-0.08}[16+1] = 17e^{-0.08}$$

$\Downarrow$

E.

17.4. **Graphical interpretation of binomial pricing of call and put options.** For simplicity, let us assume that the stock does not pay dividends in this example. The following image contains the payoff curve of the call option (the blue curve) on the same coordinate system as the payoff curve of its replicating portfolio (the orange line).



We see that the two graphs intersect precisely at the two asset prices which are possible in the associated one-period binomial model. These are the only two points at which the two curves coincide. This is sufficient to claim that the replicating portfolio is, indeed, replicating for the call in the given model since the two possible stock prices signify the only two possible states-of-the-world.

The positive slope of the orange line indicates that the Δ in the replicating portfolio is itself positive. This means that the replicating portfolio for a call will **always** entail **purchasing** shares of stock. Moreover, the slope is necessarily smaller than one, so that the number of purchased shares of stock is always less than one.

The negative vertical-axis intercept of the orange line implies that $B < 0$. Thus, borrowing $|B|$ will always be part of the replicating portfolio for a call option.

Similarly, one can obtain that the Δ of the put option will always be between -1 and 0. So, the replicating portfolio for a put will necessarily entail short-selling the underlying. The B in the put's replicating portfolio will always be non-negative. Thus, the amount B is initially deposited to earn the continuously compounded risk-free interest rate r.

Problem 17.3. Consider the straddle in Problem 17.2.

- Draw the payoff curve of the straddle.
- In the same coordinate system, draw the payoff curve of the straddle's replicating portfolio in the given one-period binomial model for the stock price.
- Find the Δ and the B for the replicating portfolio.
- Precisely describe the investment in the risky asset and the risk-less investment in the replicating portfolio.

Solution: