Options on Currency/Exchange Rate

- Underlying asset ... "foreign currency" FC
- Asset price ... exchange rate from the FC to the "domestic" currency DC
  
  \[
  \begin{array}{c|c}
  \hline
  \text{DC} & \text{FC} \\
  \hline
  0.01 \$/\¥ & \text{e.g.} \\
  \hline
  \end{array}
  \]

  1 ¥ is worth $0.01.

Notation: \( x(t), t \geq 0 \)

↑ the exchange rate from FC to DC at time-\( t \)

Example. Simple investment in FC.

@ time-\( 0 \):

- **Buy** 1 unit of FC
- → the initial cost in DC is \( x(0) \)
- **Deposit** the 1 unit of FC @ the "foreign" cont.-comp. risk-free interest rate \( r_F \)

@ time-\( T \):

- **Withdraw** the balance, i.e., \( e^{r_F T} \) units of FC
- **Exchange** this amt back to DC;
  
  \[
  \text{get: } x(T)e^{r_F T}
  \]

  **rnd var.**
Analogy:
cont-div-paying stocks $\iff$ "foreign" currency
\[ \delta \iff r_f \]
(dividend yield)

Example:
- Prepaid forward price:
  \[ F_{0,T}^{P}(x) = x(0) e^{-r_f T} \]
- Forward price: cont.-compounded, risk-free interest rate for DC
  \[ F_{0,T}(x) = x(0) e^{-\delta (\delta - r_f) T} \]
- European:
  - Payoff:
    - Call: \((x(T) - K)_+\)
    - Put: \((K - x(T))_+\) w/exercise date \(T\) and the strike price \(K\)
- Initial prices of the call/put
  \(V_c(0,x,K), V_p(0,x,K)\)

Put-call parity
\[ V_c(0,x,K) - V_p(0,x,K) = x(0) e^{-r_f T} - Ke^{-\delta T} \]
Special Symmetry

\[ V_c(t, x, K) \uparrow \text{strike price} \]
\[ \uparrow \text{valuation date} \quad (0 \text{ or } T) \]
\[ \uparrow \text{payoff} \]
\[ \text{THE PAYOFF:} \]
\[ V_c(T, x, K) = (x(T) - K)_+ \]
\[ = x(T) \cdot K \cdot (\frac{1}{K} - \frac{1}{x(T)})_+ \]
\[ \text{exchange rate } FC \rightarrow DC \text{ at time } T \]

\[ V_c(T, x, K) = x(T) \cdot K \cdot V_p(T, \frac{1}{x}, \frac{1}{K}) \]

\[ \downarrow \text{no arbitrage} \]

\[ V_c(0, x, K) = x(0) \cdot K \cdot V_p(0, \frac{1}{x}, \frac{1}{K}) \]

\[ \text{Put-call symmetry/duality} \]
\[ V_p(0, x, K) = x(0) \cdot K \cdot V_c(0, \frac{1}{x}, \frac{1}{K}) \]