Properties of European-option prices

OCT 23rd, 2015

No Arbitrage $\Rightarrow$ $\left\{ \begin{array}{l} \text{parity} \\ \text{symmetry} \\ \text{algebraic equalities in prices} \end{array} \right.$

$\boxed{\text{EQUALITY}}$

$\Downarrow$

? Inequalities ?

$\swarrow$ bounds on option prices

$\searrow$ properties of call/put-prices as functions of the strike

- $\rightarrow$ monotonicity
- $\rightarrow$ bounds on the "slope"
- $\rightarrow$ convexity

OCT 23RD (PM)

UNIVERSITY OF TEXAS AT AUSTIN

Lecture 14

Bounds on option prices. Bull spreads

14.1. **Introduction.** As was the case with put-call parity, we can discover further properties of option prices stemming from the no-arbitrage assumption only. These properties are not going to be equalities (as was the case with put-call parity), but rather they will be inequalities which we can classify into three different categories. The first category is a set of upper and lower bounds one can obtain on European call and put prices by simply using the put-call parity and the payoff structure. The second and third categories rely on the analysis of vanilla option prices as functions of the strike price with all the other parameters/arguments fixed. Here, we will incounter monotonicty and convexity inequalities.

14.2. **Bounds on option prices.** To ensure we get the most general statements we can, let us denote by $V_C(t)$ and $V_P(t)$ the values of European calls and puts at time $t \in [0, T]$, respectively. We temporarily fix the underlying asset S, their common exercise date T and strike price K.

Due to put-call parity, which is valid at any time t, and not merely at time 0 (at which we usually use it!), we have

$$V_C(t) - \overbrace{V_P(t)}^{\geq 0} = F^P_{t,T}(S) - Ke^{-r(T-t)}.$$

Since both the put- and the call-option prices must be nonnegative at all times, we conclude that

$$V_C(t) \geq (F^P_{t,T}(S) - Ke^{-r(T-t)})_+ = \max(F^P_{t,T}(S) - Ke^{-r(T-t)}, 0)$$

On the other hand, trivially,

$$S(T) \geq (S(T) - K)_+ = V_C(T).$$

So, $F^P_{t,T}(S) \geq V_C(t)$, and

$$F^P_{t,T}(S) \geq V_C(t) \geq (F^P_{t,T}(S) - Ke^{-r(T-t)})_+.$$

14.3. **Monotonicity inequalities.** Let us temporarily fix the underlying asset and the exercise date T. Consider two strike prices $K_1 < K_2$. Looking at the payoffs of the two calls with these strikes, we see that the payoff of the K_1-call dominates the payoff of the K_2-call.

Put-price bounds

- Put-call parity:

$$V_p(t) - \underbrace{V_c(t)}_{\geq 0} = \underbrace{Ke^{-r(T-t)} - F^P_{t,T}(S)}$$

$$V_p(t) \geq \begin{cases} 0 \\ Ke^{-r(T-t)} - F^P_{t,T}(S) \end{cases}$$

$$V_p(t) \geq \left(Ke^{-r(T-t)} - F^P_{t,T}(S)\right)_+$$ Lower bd

- $Ke^{-r(T-t)} \geq V_p(t)$ Upper bd.

Call/put prices as functions of the strike

- All other inputs, but the strike, are fixed.
- Set $t=0$.
- Temporarily change the notation:

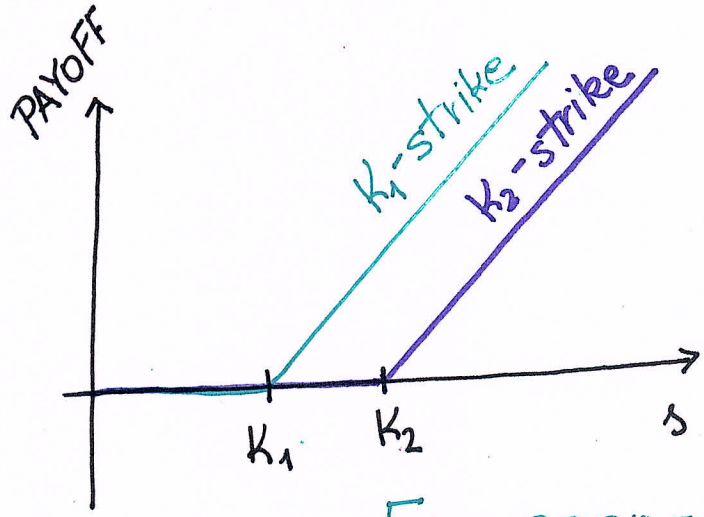
$$\left.\begin{array}{l} V_c(K) \ldots \text{call} \\ V_p(K) \ldots \text{put} \end{array}\right\} \ldots \text{prices w/ strike price } K$$

Monotonicity.

Q: Are call prices increasing/decreasing as functions of the strike?

$$\boxed{K_1 < K_2} \overset{?}{\Longrightarrow} V_c(K_1) \; ? \; V_c(K_2)$$

$$\leq \quad = \quad \geq$$



$$\Longrightarrow \boxed{V_c(K_1) \geq V_c(K_2)}$$

European-call prices are DECREASING w/ respect to the strike.

Example. Assume, to the contrary, that there exist $K_1 < K_2$ such that $\boxed{V_c(K_1) < V_c(K_2)}$.

I. ✓

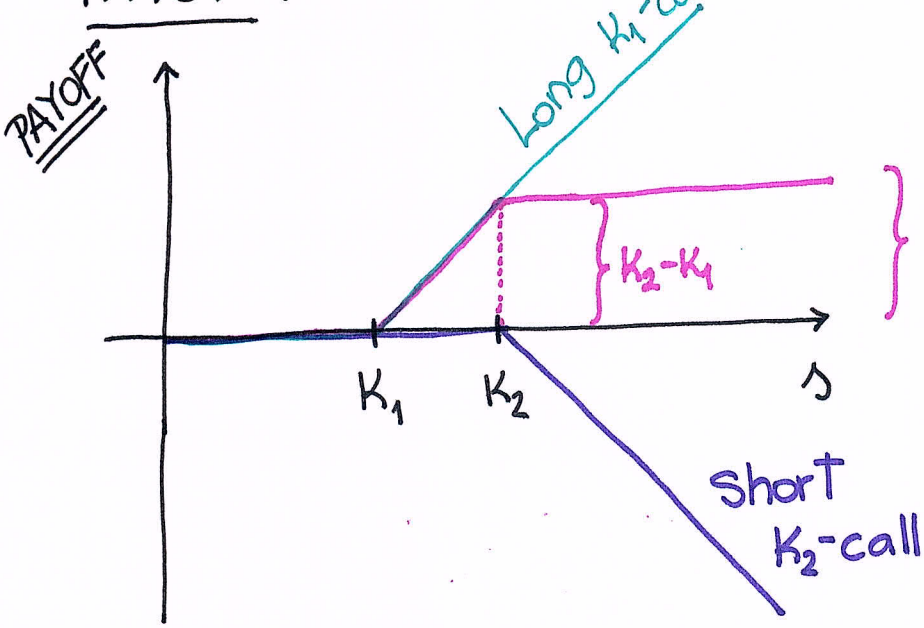
II. Propose an <u>arbitrage portfolio</u>:

- <u>LONG</u> the (relatively cheap) K_1-strike call
- <u>WRITE</u> the (relatively expensive) K_2-strike call

III. <u>Verification</u>.

<u>Initial cost</u>: $V_c(K_1) - V_c(K_2) < 0$ Initial inflow of money!

<u>PAYOFF</u>: $(S(T)-K_1)_+ - (S(T)-K_2)_+ = \cdots$



$\Downarrow$

PROFIT > 0

$\Downarrow$

This is, indeed, an arbitrage portfolio.

(CALL)

<u>BULL SPREAD</u>

<u>Note</u>: This is a <u>long</u> position w.r.t. the underlying.
$\Rightarrow$ A good <u>hedge</u> for an inherent short position.

Problem. Observe: $V_c(50) = 9$, $V_c(60) = 10$

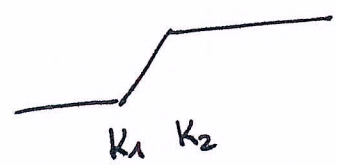
Q: Is there an arbitrage opportunity?
If so, find an arbitrage portfolio?

Q: What is the minimum arbitrage profit for the above prices?

a)
b)
c)
d)
e) No arbitrage opp.

Bull Spreads.

- Calls ✓
- Puts ✓



- LONG the K_1-strike put
- SHORT the K_2-strike put

} Put bull spread

. Draw the payoff curve.
. Calculate the profits of both the call bullspread & the put bull spread.

. Compare the two profits.

$\boxed{\text{Fact}}$ Put prices are increasing w/ respect to the strike.