Properties of European option prices

No Arbitrage \implies \begin{cases} \text{parity} \\
\text{symmetry} \\
\text{algebraic equalities in prices} \end{cases}

\begin{align*}
\text{EQUA} & \text{LITY} \\
\text{bounds on option prices}
\end{align*}

\begin{align*}
\text{properties of call/put prices as functions of the strike} \\
\rightarrow \text{monotonicity} \\
\rightarrow \text{bounds on the "slope"} \\
\rightarrow \text{convexity}
\end{align*}
14.1. **Introduction.** As was the case with put-call parity, we can discover further properties of option prices stemming from the no-arbitrage assumption only. These properties are not going to be equalities (as was the case with put-call parity), but rather they will be inequalities which we can classify into three different categories. The first category is a set of upper and lower bounds one can obtain on European call and put prices by simply using the put-call parity and the payoff structure. The second and third categories rely on the analysis of vanilla option prices as functions of the strike price with all the other parameters/arguments fixed. Here, we will encounter monotonicity and convexity inequalities.

14.2. **Bounds on option prices.** To ensure we get the most general statements we can, let us denote by $V_C(t)$ and $V_P(t)$ the values of European calls and puts at time $t \in [0,T]$, respectively. We temporarily fix the underlying asset $S$, their common exercise date $T$ and strike price $K$.

Due to put-call parity, which is valid at any time $t$, and not merely at time 0 (at which we usually use it!), we have

$$V_C(t) - V_P(t) = F_{S,T}^C(S) - K e^{-r(T-t)}.$$

Since both the put- and the call-option prices must be nonnegative at all times, we conclude that

$$V_C(t) \geq \max(F_{t,T}^P(S)-K e^{-r(T-t)}; 0) = \max\left(F_{t,T}^P(S) - Ke^{-r(T-t)}; 0\right).$$

On the other hand, trivially,

$$S(T) \geq (S(T) - K)_{+} = V_C(T).$$

So, $F_{t,T}^P(S) \geq V_C(t)$, and

$$F_{t,T}^P(S) \geq V_C(t) \geq (F_{t,T}^P(S) - Ke^{-r(T-t)})_{+}.$$

14.3. **Monotonicity inequalities.** Let us temporarily fix the underlying asset and the exercise date $T$. Consider two strike prices $K_1 < K_2$. Looking at the payoffs of the two calls with these strikes, we see that the payoff of the $K_1$—call dominates the payoff of the $K_2$—call.
Put-price bounds

- Put-call parity:
  \[ V_p(t) - V_c(t) = K e^{-r(t-t)} - F_{t,T}^p(S) \]
  \[ \geq 0 \]

- \[ V_p(t) \geq \begin{cases} 
0 & \\
K e^{-r(t-t)} - F_{t,T}^p(S) & 
\end{cases} \]

- \[ V_p(t) \geq (K e^{-r(t-t)} - F_{t,T}^p(S))_+ \] Lower bd

- \[ K e^{-r(t-t)} \geq V_p(t) \] Upper bd
Call/put prices as functions of the strike

- All other inputs, but the strike, are fixed.
- Set $t=0$.
- Temporarily change the notation:
  
  \[
  V_c(K) \text{... call} \quad \text{prices w/ strike price } K \\
  V_p(K) \text{... put}
  \]

**Monotonicity.**

Q: Are call prices increasing/decreasing as functions of the strike?

$$K_1 < K_2 \implies V_c(K_1) \leq V_c(K_2)$$

European-call prices are **DECREASING** w/respect to the strike.
Example. Assume, to the contrary, that there exist \( k_1 < k_2 \) such that \( V_c(k_1) < V_c(k_2) \).

I. √

II. Propose an arbitrage portfolio:
   \[
   \begin{align*}
   &\text{LONG} & \text{the (relatively cheap) } k_1\text{-strike call} \\
   &\text{WRITE} & \text{the (relatively expensive) } k_2\text{-strike call}
   \end{align*}
   \]

II. Verification.
   Initial cost: \( V_c(k_1) - V_c(k_2) < 0 \)
   Payoff: \( (S(T) - k_1)_+ - (S(T) - k_2)_+ = \cdots \)

Note: This is a **long** position w.r.t. the underlying. ⇒ A good hedge for an inherent short position.

This is, indeed, an arbitrage portfolio. (CALL) **BULL SPREAD**
Problem: Observe: $V_c(50) = 9$, $V_c(60) = 10$

Q: Is there an arbitrage opportunity? If so, find an arbitrage portfolio?

Q: What is the minimum arbitrage profit for the above prices?

a) 

b) 

c) 

d) 

e) No arbitrage opp.

Bull Spreads: 
- Calls ✓ 
- Puts ✓

\[ K_1 \quad K_2 \]

\[ \begin{cases} \text{LONG} \quad \text{the } K_1 \text{-strike put} \\ \text{SHORT} \quad \text{the } K_2 \text{-strike put} \end{cases} \]

Put bull spread

Put prices are increasing w.r.t. respect to the strike. 

Draw the payoff curve.

Calculate the profits of both the call bullspread & the put bull spread.

Compare the two profits.