Properties of European put/call prices [Section 9.3 in McDonald]

Recall: No arbitrage $\Rightarrow$ put-call parity

$\boxed{\text{an equality}}$

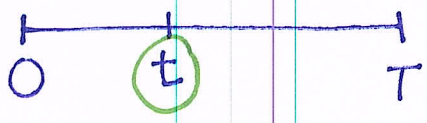
Now: $\boxed{\text{Inequalities}}$ ... consequences of the NO-ARBITRAGE assumption only

- bounds
- monotonicity
  - $\downarrow$ bounds on the "slope" of increase/decrease
  - $\downarrow$ convexity

we see option prices as functions of the STRIKE (w/ all else fixed)

Bounds

$S(t)$ ... the underlying's price



$\left.\begin{array}{l} V_c(t) \\ V_p(t) \end{array}\right\}$ .... value/PRICE of a European $\left\{\begin{array}{l}\text{call}\\ \text{put}\end{array}\right\}$ @ time t

T ... exercise date
K ... the strike price

$\boxed{\text{CALLS}}$ upper bd $\geq V_c(t) \geq$ lower bd

<u>Lower bd:</u>

- $V_c(t) \geq 0$
- <u>Put-call parity</u>

$$\underline{\underline{V_c(t)}} \geq V_c(t) - \underbrace{V_p(t)}_{\geq 0} = \underline{\underline{F^P_{t,T}(S) - PV_{t,T}(K)}}$$

$$V_c(t) \geq \text{MAX}\left[0, F^P_{t,T}(S) - PV_{t,T}(K)\right]$$

Both terms are necessary!

<u>Note:</u> ~~If~~ $F^P_{t,T}(S) - PV_{t,T}(K) \geq 0$

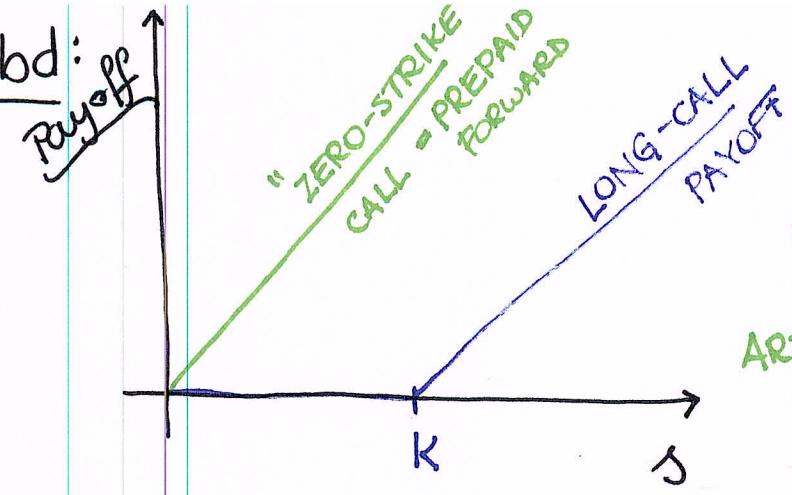
$$\Leftrightarrow \quad F^P_{t,T}(S) \geq PV_{t,T}(K)$$

$$\Leftrightarrow \quad \underbrace{FV_{t,T}\left(F^P_{t,T}(S)\right)} \geq K$$

$$\Leftrightarrow \quad \underbrace{F_{t,T}(S) \geq K}.$$

The strike K can be chosen as bigger than the forward price!

Upper bd:



$\Longrightarrow F^P_{t,T}(S) \geq V_c(t)$

NO ARBITRAGE!

$\boxed{\text{PUTS}}$

$$Ke^{-r(T-t)} \geq V_p(t) \geq \text{MAX}\left[0;\ PV_{t,T}(K) - F^P_{t,T}(S)\right]$$

The best scenario for the put's owner is that the underlying becomes worthless.

PRICES AS FUNCTIONS of the strike K

All other inputs are fixed.

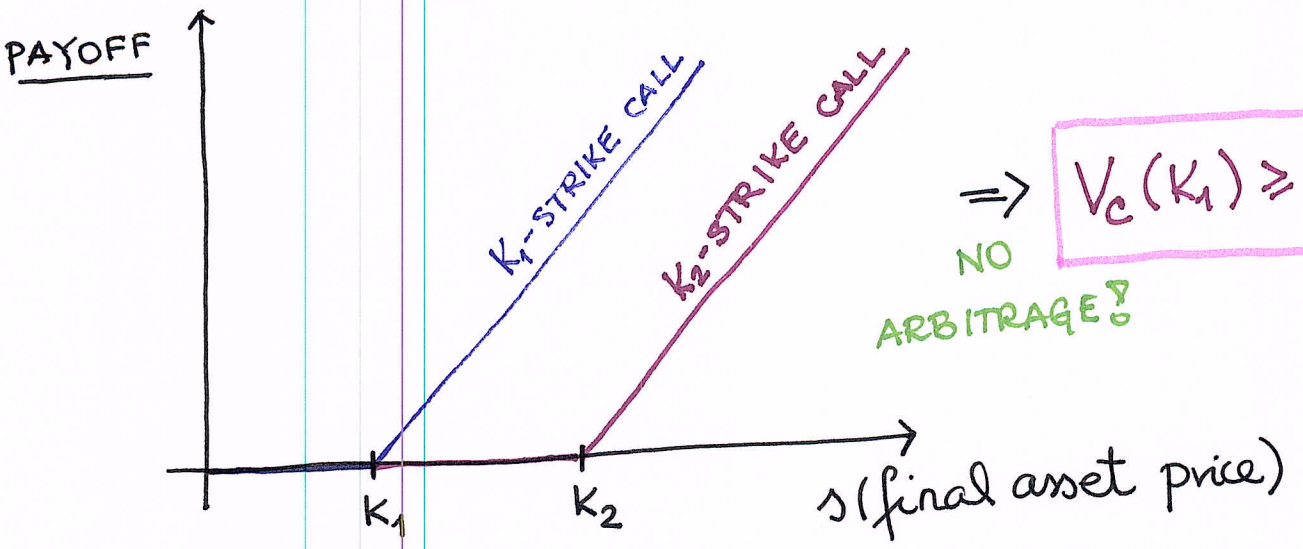
Set $t=0$.

$$\left.\begin{array}{l} V_c(K) \ \ldots \text{ call} \\ V_p(K) \ \ldots \text{ put} \end{array}\right\} \text{prices w/ strike price } K$$

Monotonicity.

$\boxed{\text{CALLS}}$ $K_1 < K_2$

Q: $V_c(K_1) \overset{?}{\leq} V_c(K_2)$



$\Rightarrow \boxed{V_c(K_1) \geq V_c(K_2)}$

NO ARBITRAGE!

European-call prices are DECREASING with respect to the strike.

Example. Assume TO THE CONTRARY that there exist $K_1 < K_2$ such that $\boxed{V_c(K_1) < V_c(K_2)}$.

I. Suspicion of arbitrage.

II. Propose an arbitrage portfolio:

- LONG the relatively cheap K_1-strike call
- SHORT the —"— expensive K_2-strike call

III. Verification.

Initial cost: $V_c(K_1) - V_c(K_2) < 0$

$\Downarrow$

Inflow!

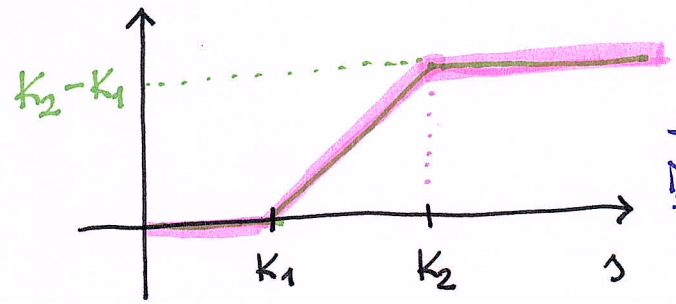
Payoff: $(S(T)-K_1)_+ - (S(T)-K_2)_+ \geq 0$

PROFIT > 0

$\Downarrow$

An arbitrage portfolio.

THE (CALL) BULL SPREAD

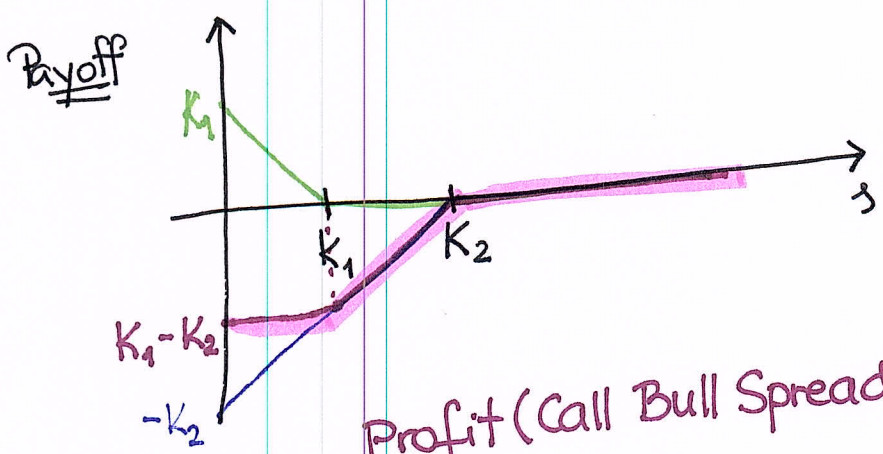


Note: LONG w/ respect to the underlying: HEDGE long short position.

Q : Construct a PUT bull spread!

$\boxed{K_1 < K_2}$ $\left.\begin{array}{l} K_1\text{-strike put} \\ K_2\text{-strike put} \end{array}\right\}$ otherwise identical

(*) $\begin{cases} \bullet \text{ LONG the } \boxed{K_1\text{-strike put}} \\ \bullet \text{ SHORT the } \boxed{K_2\text{-strike put}} \end{cases}$



NEVER POSITIVE!

Profit(Call Bull Spread) = Profit(Put Bull Spread)

PUT-CALL PARITY

$\boxed{\text{PUT}}$ Claim: Put prices are INCREASING as functions of the strike. Show this!