Properties of European put/call prices

Recall: No arbitrage \implies put-call parity an equality

Now: Inequalities ... consequences of the NO-ARBITRAGE assumption only

bounds

monotonicity

bounds on the "slope" of increase/decrease

convexity

we see option prices as functions of the STRIKE (w/ all else fixed)
Bounds

\[ S(t) \ldots \text{the underlying's price} \]

\[ V_c(t), V_p(t) \] \ldots value/price of a European \{ call, put \} at \text{time} T

\[ T \ldots \text{exercise date} \]
\[ K \ldots \text{the strike price} \]

**CALLS** upper bd \geq V_c(t) \geq lower bd

**Lower bd:**
- \( V_c(t) \geq 0 \)
- **Put-call parity**
  \[ V_c(t) \geq V_c(t) - V_p(t) = F_{t,T}^p(S) - PV_{t,T}(K) \geq 0 \]

**Note:** If \( F_{t,T}^p(S) - PV_{t,T}(K) \geq 0 \)

\[ \iff F_{t,T}^p(S) \geq PV_{t,T}(K) \]
\[ \iff F_{t,T}^p \left( F_{t,T}^p(S) \right) \geq K \]
\[ \iff F_{t,T}^p(S) \geq K \]

The strike \( K \) can be chosen as bigger than the forward price.
PUTS

\[ K e^{-r(T-t)} \geq V_p(t) \geq \max \left[ 0; PV_{t,T}(K) - F_{t,T}^p(S) \right] \]

The best scenario for the put's owner is that the underlying becomes worthless.

PRICES AS FUNCTIONS of the strike $K$

All other inputs are fixed.
Set $t=0$.
\[ V_c(K) \quad \text{call} \quad V_p(K) \quad \text{put} \]

prices w/ strike price $K$

Monotonicity.

CALLS
$K_1 < K_2$

Q: \[ V_c(K_1) \quad ? \quad V_c(K_2) \]

\[ \Rightarrow F_{t,T}^p(S) \geq V_c(t) \]

\text{NO ARBITRAGE}_p
European-call prices are decreasing with respect to the strike.

Example. Assume to the contrary that there exist \( K_1 < K_2 \) such that \( V_c(K_1) < V_c(K_2) \).

I. Suspicion of arbitrage.

II. Propose an arbitrage portfolio:
- LONG the relatively cheap \( K_1 \)-strike call
- SHORT the -\( K_2 \)-strike call

III. Verification.

Initial cost: \( V_c(K_1) - V_c(K_2) < 0 \)

\[ \text{Payoff: } (S(T) - K_1)_+ - (S(T) - K_2)_+ \geq 0 \]

Note: LONG w/ respect to the underlying. HEDGE a naked position.

An arbitrage portfolio.
Q: Construct a put bull spread?

\[ K_1 < K_2 \]

\[ \begin{cases} \text{\underline{LONG the } K_1\text{-strike put} } \\
\text{\underline{SHORT the } K_2\text{-strike put} } \end{cases} \] otherwise identical

\[ \text{NEVER POSITIVE} \]

\[ \text{Profit (Call Bull Spread)} = \text{Profit (Put Bull Spread)} \]

\[ \text{PUT-CALL PARITY} \]

Claim: Put prices are INCREASING as functions of the strike. Show this?