SLOPE INEQUALITIES

We know:

\[ K_1 < K_2 \Rightarrow \begin{cases} V_c(K_1) - V_c(K_2) > 0 \text{ decreasing} \\ V_p(K_2) - V_p(K_1) \leq 0 \text{ increasing} \end{cases} \]

There is an upper bound on the change in the price:

\[ \begin{cases} V_c(K_1) - V_c(K_2) \\ V_p(K_2) - V_p(K_1) \end{cases} \leq PV_{0,T}(K_2-K_1) \]

Example. Assume to the contrary that there are \( K_1 < K_2 \) such that

\[ V_p(K_2) - V_p(K_1) > PV_{0,T}(K_2-K_1) \]

Propose an arbitrage portfolio:

\[ \{ \begin{align*} &\text{LONG the } K_1\text{-put} \\ &\text{WRITE the } K_2\text{-put} \end{align*} \] i.e., ACQUIRE a PUT BULL SPREAD.

Initial cost:

\[ V_p(K_1) - V_p(K_2) \leq PV_{0,T}(K_2-K_1) = PV_{0,T}(K_1-K_2) \]

Payoff:

\[ (K_1 - S(T))_+ - (K_2 - S(T))_+ \geq K_1 - K_2 \]

Profit:

\[ \text{PAYOFF} - \text{FV(Init. Cost)} > K_1 - K_2 + \text{FV}(PV(K_2 - K_1)) \]

\[ = 0 \]

\[ \Rightarrow \text{(strong) arbitrage} . \]
12. You are given:

(i) \( C(K, T) \) denotes the current price of a \( K \)-strike \( T \)-year European call option on a non-dividend-paying stock.
\[ \Rightarrow F_{0,T}^P (S) = S(0) \]

(ii) \( P(K, T) \) denotes the current price of a \( K \)-strike \( T \)-year European put option on the same stock.

(iii) \( S \) denotes the current price of the stock.

(iv) The continuously compounded risk-free interest rate is \( r \).

Which of the following is (are) correct?

(I) \[ 0 \leq C(50, T) - C(55, T) \leq 5e^{-rT} \]
\[ \Rightarrow \text{TRUE} \Rightarrow (\, \checkmark \, ) \]

(II) \[ 50e^{-rT} \leq P(45, T) - C(50, T) + S \leq 55e^{-rT} \]

(III) \[ 45e^{-rT} \leq P(45, T) - C(50, T) + S \leq 50e^{-rT} \]

(A) (I) only

(B) (II) only

(C) (III) only

(D) (I) and (II) only

(E) (I) and (III) only

\[ V_p(45) - V_c(50) + S(0) \]

\[ V_c(50) = \sqrt{V_p(50) + S(0) - 50e^{-rT}} \]

\[ 45e^{-rT} \leq V_p(45) - V_p(50) + 50e^{-rT} \leq 50e^{-rT} \Rightarrow \text{III correct} \]

- \( (50 - 45)e^{-rT} \)

- \( - (50 - 45)e^{-rT} \)

"slope" inequality

increase of put prices

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\[ K_1 < K_2 \Rightarrow V_C(K_1) - V_C(K_2) \leq PV_{0,T}(K_2 - K_1) \]

**Example.** Assume to the contrary that there are \( K_1 < K_2 \) such that

\[ V_C(K_1) - V_C(K_2) > PV_{0,T}(K_2 - K_1) \]

Propose an arbitrage portfolio:

- **SHORT** \( K_1 \)-strike call
- **LONG** \( K_2 \)-strike call

**Payoff:**

\[ -(S(T) - K_1)_+ + (S(T) - K_2)_+ = \]

\[
\begin{cases} 
\emptyset & \text{if } S(T) \leq K_1 \\
-(S(T) - K_1) = K_1 - S(T) & \text{if } K_1 < S(T) \leq K_2 \\
-(S(T) - K_1) + (S(T) - K_2) = K_1 - K_2 & \text{if } K_2 < S(T)
\end{cases}
\]
INITIAL COST: \[ V_c(K_2) - V_c(K_1) < -PV(K_2-K_1) \]

**PROFIT** = \( \text{PAYOFF} - FV(\text{INIT. COST}) \) \( \geq K_1 - K_2 \) \( > FV(PV(K_2-K_1)) \) \( \Rightarrow \text{ARBITRAGE?} \)

**Note:** The BEAR spread is **short** with respect to the underlying. So, it's an appropriate HEDGE for a **LONG** position w/ respect to the underlying.

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**Put Bear spread.**

**PAYOFF**

\[ K_2 - K_1 \]

\( K_1 \) \( K_2 \)

\( \begin{cases} \text{Long } K_2\text{-strike put} \\ \text{Short } K_1\text{-strike put} \end{cases} \)

**PUT-CALL PARITY**

\[ \text{PROFIT} (\text{Put bear spread}) = \text{PROFIT} (\text{Call bear spread}) \]

**Q1:** For the violation of which one of the above inequalities would you use the **PUT BEAR SPREAD** as an arbitrage portfolio?