SLOPE INEQUALITIES

We know:

\[ K_1 < K_2 \Rightarrow \left\{ \begin{array}{l} V_c(K_1) - V_c(K_2) \geq 0 \\ V_p(K_2) - V_p(K_1) \geq 0 \end{array} \right. \]

There is an upper bound on the change in the price:

\[ \left\{ \begin{array}{l} V_c(K_1) - V_c(K_2) \\ V_p(K_2) - V_p(K_1) \end{array} \right\} \leq PV_{0,T}(K_2 - K_1) \]

**Example.** Assume to the contrary that there are

\[ K_1 < K_2 \text{ such that } V_p(K_2) - V_p(K_1) > PV_{0,T}(K_2 - K_1) \]

Propose an arbitrage portfolio:

\[ \{ \begin{array}{l} \text{long the } K_1\text{-put} \\ \text{write the } K_2\text{-put} \end{array} \} \]

i.e., acquire a put bull spread.

Initial cost: \[ V_p(K_1) - V_p(K_2) < -PV_{0,T}(K_2 - K_1) = PV_{0,T}(K_1 - K_2) \]

Payoff: \[ (K_1 - S(T))^+ - (K_2 - S(T))^+ \geq K_1 - K_2 \]

Profit: \[ \text{payoff} - FV(\text{Init. cost}) = K_1 - K_2 + FV(PV(K_2 - K_1)) = 0 \]

\[ \Rightarrow \text{(strong) arbitrage}. \]
There is also an upper bound on the absolute change in the option price as the strike shifts:

\[ K_1 < K_2 \Rightarrow \begin{cases} V_c(K_1) - V_c(K_2) \\ V_p(K_2) - V_p(K_1) \end{cases} \leq PV_{0,T}(K_2-K_1) \]

**Example.** Assume to the contrary that there are \( K_1 < K_2 \) such that

\[ V_c(K_1) - V_c(K_2) > PV_{0,T}(K_2-K_1) \]

Propose an arbitrage portfolio:

\( \{ \)

\( \cdot \) Short/write the \( K_1 \)-strike call

\( \cdot \) Long the \( K_2 \)-strike call

\( \} \)

**INIT. COST:** \( V_c(K_2) - V_c(K_1) < -PV(K_2-K_1) \)

**PAYOFF:**

\[ -(S(T) - K_1)_+ + (S(T) - K_2)_+ = \begin{cases} -(S(T) - K_1) = K_1 - S(T) & \text{if } S(T) \leq K_1 \\ -(S(T) - K_1) + (S(T) - K_2) - (K_4 - K_2) & \text{if } K_1 < S(T) \leq K_2 \\ & \text{if } K_2 < S(T) \end{cases} \]

\[ \Rightarrow \text{PAYOFF} > K_4 - K_2 \]

**Diagram**

- **Call Bear Spread**
  - Payoff
  - Initial Cost
  - Diagram showing the payoffs for different scenarios of \( S(T) \) relative to \( K_1 \) and \( K_2 \).
\[
\text{PROFIT} = \text{PAYOFF} - \text{FV} (\text{Init. Cost}) > 0 \Rightarrow \text{ARBITRAGE}\
\Rightarrow K_1 - K_2 \Rightarrow \text{FV}(\text{PV}(K_2 - K_1)) = K_2 - K_1
\]

*Interesting arbitrage portfolio*

\text{SHORT} with respect to the underlying?

\Rightarrow \text{speculation (if you believe prices would drop (but not too much))}

\Rightarrow \text{HEDGING} for agents w/ an inherent LONG position w/ respect to the underlying.
12. You are given:

(i) \( C(K, T) \) denotes the current price of a \( K \)-strike \( T \)-year European call option on a
nondividend-paying stock. \( F^P_{0,T}(S) = S(0) \)

(ii) \( P(K, T) \) denotes the current price of a \( K \)-strike \( T \)-year European put option on the
same stock.

(iii) \( S \) denotes the current price of the stock.

(iv) The continuously compounded risk-free interest rate is \( r \).

Which of the following is (are) correct?

(I) \( 0 \leq C(50, T) - C(55, T) \leq 5e^{-rT} \) \( \text{TRUE } \Rightarrow \) \( \times, \times \)

(II) \( 50e^{-rT} \leq P(45, T) - C(50, T) + S \leq 55e^{-rT} \)

(III) \( 45e^{-rT} \leq P(45, T) - C(50, T) + S \leq 50e^{-rT} \)

(A) (I) only

(B) (II) only

(C) (III) only

(D) (I) and (II) only

(E) (I) and (III) only

\[ V_P(45) = V_C(45) - V_C(50) + S(0) \]

\[ V_C(45) + 45e^{-rT} - S(0) - V_C(50) + S(0) \]

\[ 45e^{-rT} \leq V_C(45) - V_C(50) + 45e^{-rT} \leq 50e^{-rT} \]

\( \text{PUT-CALL PARITY: } \)

\( \Rightarrow \) \( \boxed{E} \)
\[ P \text{UT BEAR SPREAD} \]

\[
\begin{align*}
\text{Payoff} & \quad (K_2 - K_1) \\
\text{profit} & \quad \text{PUT-CALL PARITY} \\
\text{PROFIT(Call Bear Spread)} & = \text{PROFIT(Put Bear Spread)} \\
\text{PAYOFF (Call Bear Spread)} - FV \left( V_c (K_2) - V_c (K_1) \right) & = \\
= \text{PAYOFF (Put Bear Spread)} - (K_2 - K_1) \\
- FV \left( V_p (K_2) + F_{o,t}^p (S) - PV(K_2) - (V_p (K_1) + F_{o,t}^p (S) - PV(K_1)) \right) & = \\
= \text{PAYOFF (Put Bear Spread)} - (K_2 - K_1) \\
- FV \left( V_p (K_2) - V_p (K_1) \right) - FV \left( -PV(K_2) + PV(K_1) \right) & = \\
- \left( -K_2 + K_1 \right) \\
= K_2 - K_1 \\
\end{align*}
\]

Q.1: For the violation of which one of the monotonicity and slope inequalities would you use the PUT BEAR SPREAD as an arbitrage portfolio?

The put-price increase.