SLOPE INEQUALITIES

We know:

$$K_1 < K_2 \implies \left.\begin{array}{l} V_c(K_1) - V_c(K_2) \\ V_p(K_2) - V_p(K_1) \end{array}\right\} \geq 0 \quad \begin{array}{l}\text{decreasing} \\ \text{increasing}\end{array}$$

There is an <u>UPPER BOUND</u> ON THE change in the price.

$$K_1 < K_2 \implies \boxed{\left.\begin{array}{l} V_c(K_1) - V_c(K_2) \\ V_p(K_2) - V_p(K_1) \end{array}\right\} \leq PV_{0,T}(K_2 - K_1)}$$

<u>Example</u>. Assume TO THE CONTRARY that there are $K_1 < K_2$ such that

$$V_p(K_2) - V_p(K_1) > PV_{0,T}(K_2 - K_1)$$

Propose an <u>arbitrage portfolio</u>:

- LONG the K_1-put
- WRITE the K_2-put

i.e., ACQUIRE a PUT BULL SPREAD.

<u>Initial cost</u>: $V_p(K_1) - V_p(K_2) < -PV_{0,T}(K_2 - K_1) = PV_{0,T}(K_1 - K_2)$

<u>PAYOFF</u>: $(K_1 - S(T))_+ - (K_2 - S(T))_+ \geq K_1 - K_2$

<u>PROFIT</u>: PAYOFF $-$ FV(Init. Cost) $> K_1 - K_2 + FV(PV(K_2 - K_1))$

$= 0$

$\implies$ (strong) arbitrage.

There is also an UPPER BOUND on the absolute change in the option price as the strike shifts:

$$\boxed{K_1 < K_2 \Rightarrow \left.\begin{array}{l} V_C(K_1) - V_C(K_2) \\ V_P(K_2) - V_P(K_1) \end{array}\right\} \leq PV_{0,T}(K_2 - K_1)}$$

<u>Example</u>. Assume TO THE CONTRARY that there are $K_1 < K_2$ such that

$$\boxed{V_C(K_1) - V_C(K_2) > PV_{0,T}(K_2 - K_1)}$$

$$\Updownarrow$$

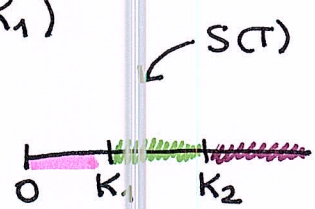
$$V_C(K_1) > V_C(K_2) + PV(K_2 - K_1)$$

Propose an arbitrage portfolio:

- SHORT/WRITE the K_1-strike call
- LONG the K_2-strike call

<u>INIT. COST</u>: $V_C(K_2) - V_C(K_1) < -PV(K_2 - K_1)$

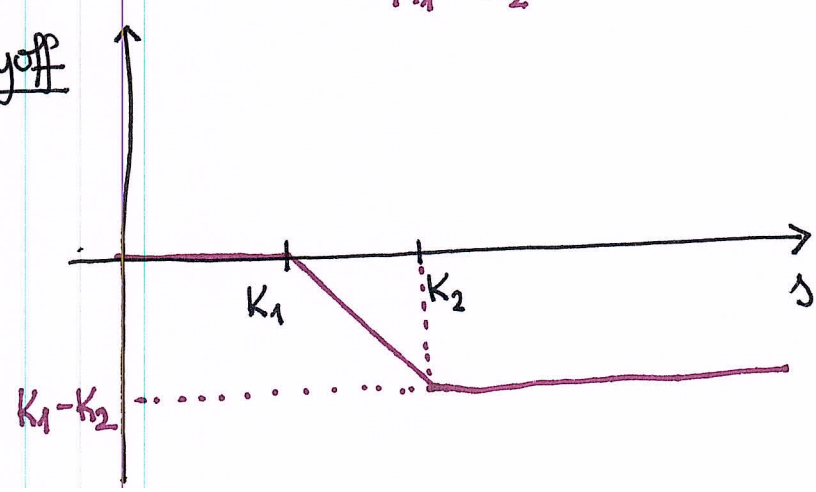
<u>PAYOFF</u>: $-(S(T) - K_1)_+ + (S(T) - K_2)_+ = ?$



$$= \begin{cases} 0 & \text{IF } S(T) \leq K_1 \\ -(S(T) - K_1) = K_1 - S(T) & \text{IF } K_1 < S(T) \leq K_2 \\ \underbrace{-(\cancel{S(T)} - K_1) + (\cancel{S(T)} - K_2)}_{K_1 - K_2} & \text{IF } K_2 < S(T) \end{cases}$$

$$\Rightarrow \boxed{\text{PAYOFF} \geq K_1 - K_2}$$

<u>Payoff</u>



CALL $\boxed{\text{BEAR}}$ SPREAD

$$\boxed{\text{PROFIT}} = \underbrace{\text{PAYOFF}}_{\geq K_1 - K_2} - \underbrace{\text{FV(Init. Cost)}}_{> FV(PV(K_2 - K_1)) = K_2 - K_1} > 0 \Rightarrow \text{ARBITRAGE!}$$

* Interesting arbitrage portfolio *

→ $\boxed{\text{SHORT}}$ with respect to the underlying!

⇒ speculation (if you believe prices would drop (but not too much))

⇒ HEDGING for agents w/ an inherent LONG position w/ respect to the underlying.

12. You are given:

(i) $C(K, T)$ denotes the current price of a K-strike T-year European call option on a nondividend-paying stock. $\rightarrow F^P_{0,T}(S) = S(0)$

(ii) $P(K, T)$ denotes the current price of a K-strike T-year European put option on the same stock.

(iii) S denotes the current price of the stock.

(iv) The continuously compounded risk-free interest rate is r.

Which of the following is (are) correct?

(I) $0 \le C(50, T) - C(55, T) \le 5e^{-rT}$ TRUE $\Rightarrow$ (A), (C)

(II) $50e^{-rT} \le P(45, T) - C(50, T) + S \le 55e^{-rT}$

(III) $45e^{-rT} \le P(45, T) - C(50, T) + S \le 50e^{-rT}$

$$\underbrace{V_p(45)} - V_c(50) + S(0)$$

PUT-CALL PARITY:

$$V_p(45) = \boxed{V_c(45) + 45e^{-rT} - S(0)}$$

(A) (I) only

(B) (II) only

(C) (III) only

(D) (I) and (II) only

(E) (I) and (III) only

$$V_c(45) + 45e^{-rT} - \cancel{S(0)} - V_c(50) + \cancel{S(0)}$$

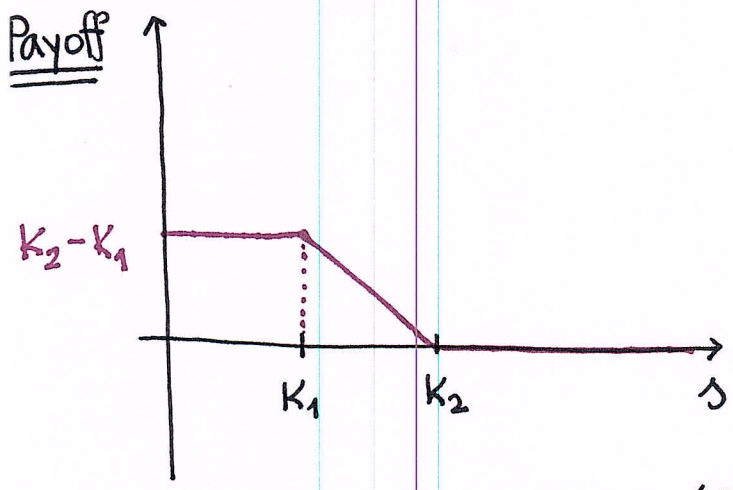
$$45e^{-rT} \le \underbrace{V_c(45) - V_c(50)}_{0 \le \quad \le 5e^{-rT}} + 45e^{-rT} \le 50e^{-rT}$$

$\Downarrow$

(III) TRUE!

$\Rightarrow \boxed{E.}$

PUT BEAR SPREAD



- LONG K_2-strike put
- SHORT K_1-strike put

Q#1:

PUT-CALL PARITY

PROFIT(Call Bear Spread) = PROFIT(Put Bear Spread)

$$\text{PAYOFF (Call Bear Spread)} - FV\left(V_c(K_2) - V_c(K_1)\right) =$$

$$= \text{PAYOFF (Put Bear Spread)} - (K_2 - K_1)$$

$$- FV\left(V_p(K_2) + \cancel{F^P_{0,T}(S)} - PV(K_2) - \left(V_p(K_1) + \cancel{F^P_{0,T}(S)} - PV(K_1)\right)\right)$$

$$= \boxed{\text{PAYOFF (Put Bear Spread)} - FV\left(V_p(K_2) - V_p(K_1)\right)} \cancel{- (K_2 - K_1)} \underbrace{- FV\left(-PV(K_2) + PV(K_1)\right)}_{\underbrace{-(-K_2 + K_1)}_{= \cancel{K_2 - K_1}}}$$

Q#2: For the violation of which one of the monotonicity and slope inequalities would you use the PUT BEAR SPREAD as an arbitrage portfolio?

The put price increase.