Def'n. Consider a European-style derivative security. A static portfolio is said to be a replicating portfolio for this derivative security if their payoffs are equal.

Q: Create a replicating portfolio for a forward contract using a call and a put on the same underlying?

The payoff of the forward: \( S(T) - F_{0,T}(S) \)

\[
\begin{align*}
\text{LONG} & \quad \text{call} \quad \text{w/ strike } K = F_{0,T}(S) \\
\text{SHORT} & \quad \text{put} \quad \text{and exercise date } T
\end{align*}
\]

What is the initial cost of this replicating portfolio? Replicates a forward
\( \Rightarrow \) has the same initial cost as a forward
\( \Rightarrow \) cost = 0.

Alternatively:

\[
V_c(0) - V_p(0) = F_{0,T}^p(S) - PV_{0,T}(K)
= F_{0,T}^p(S) - PV_{0,T}(F_{0,T}(S)) = 0
\]

Q: Does the replicating portfolio need to be unique?

Absolutely not!

\[
\begin{align*}
\text{Long prepaid forward contract w/delivery @ } T \\
\text{Borrow } F_{0,T}^d(S) @ \text{the risk-free rate to be repaid @ time-T.}
\end{align*}
\]
CHOOSER OPTIONS (As-you-like-it options)

written up
- T*
- T
- K

exercise date
* the payoff happens here*

The owner of the option decides if the option is to continue its life as a CALL or a PUT w/ strike K.

\[ V_{CH}(0, T^*, T, K) = ? \]

valuation date

First: What is the optimal rational way to choose @ time-T*?

Imagine you decide to sell the chooser option @ time-T*. What's the price you can get for it?

\[ \max \left( V_C(T^*, T, K), V_P(T^*, T, K) \right) = \]

valuation date

exercise date

\[ = V_{CH}(T^*, T^*, T, K) \]
\[ \max(a, b) = a + \max(0, b-a) = a + (b-a)_+ \]

\[ \Rightarrow \]

\[ V_{CH}(T^*, T^*_1, K) = \max(V_c(T^*, T, K), V_p(T^*, T, K)) \]

\[ = V_c(T^*, T, K) + \max(0, V_p(T^*, T, K) - V_c(T^*, T, K)) \]

\[ = V_c(T^*, T, K) + (V_p(T^*, T, K) - V_c(T^*, T, K))_+ \]

\[ = V_c(T^*, T, K) + (PV_{T^*}^T(K) - E_{T^*}^T(S))_+ \]

For simplicity, assume:

- no dividends
- cont.-comp., risk-free interest rate

\[ = V_c(T^*, T, K) + (K e^{-r(T-T^*)} - S(T^*))_+ \]

\[ \text{PAYOFF of a put w/ strike } e^{-r(T-T^*)} \text{ and exercise date } T^* \]

\[ \Rightarrow \]

A replicating portfolio for our chooser option:

\[ \left\{ \begin{array}{l}
\text{long call w/ strike } K \text{ and exercise date } T \\
\text{long put w/ strike } Ke^{-r(T-T^*)} \text{ and exercise date } T^*
\end{array} \right. \]

\[ \Rightarrow \]

\[ V_{CH}(0, T^*, T, K) = V_c(0, T, K) + V_p(0, T^*, Ke^{-r(T-T^*)}) \]
SAMPLE MFE

25. Consider a chooser option (also known as an as-you-like-it option) on a nondividend-paying stock. At time 1, its holder will choose whether it becomes a European call option or a European put option, each of which will expire at time 3 with a strike price of $100.

The chooser option price is $20 at time $t = 0$.

The stock price is $95 at time $t = 0$. Let $C(T)$ denote the price of a European call option at time $t = 0$ on the stock expiring at time $T, T > 0$, with a strike price of $100$.

You are given:

(i) The risk-free interest rate is 0.

(ii) $C(1) = 4$.

Determine $C(3)$.

\[
V_{CH}(0) = V_c(0,3,100) + V_p(0,1,100)
\]

\[
\text{Answer:} \quad V_c(0,3,100) = 20 - 4 - 5 = 11 \quad \Rightarrow \text{(E)}
\]
3. An insurance company sells single premium deferred annuity contracts with return linked to a stock index, the time-$t$ value of one unit of which is denoted by $S(t)$. The contracts offer a minimum guarantee return rate of $g\%$. At time 0, a single premium of amount $\pi$ is paid by the policyholder, and $\pi \times y\%$ is deducted by the insurance company. Thus, at the contract maturity date, $T$, the insurance company will pay the policyholder

$$\pi \times (1 - y\%) \times \text{Max}[S(T)/S(0), (1 + g\%)^T].$$

You are given the following information:

(i) The contract will mature in one year.
(ii) The minimum guarantee rate of return, $g\%$, is 3%.
(iii) Dividends are incorporated in the stock index. That is, the stock index is constructed with all stock dividends reinvested.
(iv) $S(0) = 100$.
(v) The price of a one-year European put option, with strike price of $103$, on the stock index is $15.21$.

Determine $y\%$, so that the insurance company does not make or lose money on this contract.