UNIVERSITY OF TEXAS AT AUSTIN

Problem Set #2

Multiple binomial periods (European).

Problem 2.1. The current price of a continuous-dividend-paying stock is \$100 per share. Its volatility is given to be 0.2 and its dividend yield is 0.03.

The continuously compounded, risk-free interest rate equals 0.06.

Consider a \$95-strike European put option on the above stock with nine months to expiration. Using a three-period forward binomial tree, find the price of this put option.

(a) \$2.97
(b) \$3.06
(c) \$3.59
(d) \$3.70
(e) None of the above.

Solution: (c)
The up and down factors in the above model are

$$u = e^{0.03\times 0.25+0.2\sqrt{0.25}} = 1.1135,$$
$$d = e^{0.03\times 0.25-0.2\sqrt{0.25}} = 0.9116.$$

The relevant possible stock prices at the "leaves" of the binomial tree are

$$S_{ddd} = d^3S(0) = 100(0.9116)^3 = 75.7553,$$
$$S_{ddu} = d^2uS(0) = 92.5335.$$

The remaining two final states of the world result in the put option being out-of-the-money at expiration.

The risk-neutral probability of the stock price moving up in a single period is

$$p^* = \frac{1}{1+e^{0.2\sqrt{0.25}}} = 0.475.$$

So, the price of the European put option equals

$$V_P(0) = e^{-0.06(3/4)}\left[(95-75.7553)(1-0.475)^3 + (95-92.5335)(3)(1-0.475)^2(0.475)\right] = 3.5884.$$

Problem 2.2. Let the continuously compounded interest rate be $r = 10\%$. Assume that the initial price of a non-dividend-paying stock is \$100 per share.

Consider a 5−period binomial model for the evolution of the stock price over the next year. Let $u = 1.04$ and $d = 0.96$.

(i) What is the price of a one-year, 100-strike cash call on the above asset?
Solution: The payoff function is

$$v(s) = \begin{cases} 1 & \text{if } s > 100 \\ 0 & \text{if } s \le 100 \end{cases} = \mathbb{I}_{(100,\infty)}(s)$$

So, we can get the possible payoffs of the cash call in the above tree as follows:

$$u^5S(0) = 1.04^5 \times 100 = 121.67 \Rightarrow V_0 = 1,$$
$$u^4dS(0) = 1.04^4 \times 0.96 \times 100 = 112.31 \Rightarrow V_1 = 1,$$
$$u^3d^2S(0) = 1.04^3 \times 0.96^2 \times 100 = 103.67 \Rightarrow V_2 = 1.$$

The remaining possible final stock prices are below the threshold of \$100. So, the payoffs at those final nodes are equal to zero. In fact,

$$V_{CC}(0) = e^{-rT}\mathbb{P}^*[S(T) > 100]$$

where $\mathbb{P}^*$ stands for the risk-neutral probability measure consistent with p^*. We can calculate p^* as

$$p^* = \frac{e^{(0.1/5)-0.96}}{1.04 - 0.96} = 0.7525.$$

We get

$$V_{CC}(0) = e^{-0.1}\left[(0.7525)^5 + 5(0.7525)^4(1-0.7525) + 10(0.7525)^3(1-0.7525)^2\right] = 0.8135.$$

(ii) What is the price of a one-year, at-the-money European call on the above asset?
Solution: Using the final stock prices we calculate above we get these possible payoffs at the uppermost final nodes:

$$V_0^C = 21.67, V_1^C = 12.31, V_2^C = 3.67.$$

The remaining payoffs are all zero as the call goes unexercised. The call price is

$$\begin{aligned} V_C(0) &= e^{-0.1}[21.67 \times 0.7525^5 + 12.31 \times 5(0.7525)^4(1-0.7525) + 3.67 \times 10(0.7525)^3(1-0.7525)^2] \\ &= 10.0176. \end{aligned}$$

Problem 2.3. Let the continuously compounded interest rate be $r = 11\%$ per annum. Assume that the initial price of a non-dividend-paying stock is \$100 per share.

Consider an $11-$period binomial model for the evolution of the stock price over the next year. Let $u = 1.04$ and $d = 0.96$.

What is the price of a one-year, 130-strike cash call on the above asset?

Solution: Calculating the possible final stock prices is inefficient. What we do is find the "critical" value of the possible down-steps that the stock price can take as it moves through the tree. At this number of down-steps, the payoff is still one, if there is even one more step down in the stock's path through the tree, the payoff is zero. Symbolically, we are looking for the largest $k = 0, \ldots, 11$ such that

$$u^{n-k}d^kS(0) > K \quad \Leftrightarrow \quad \left(\frac{d}{u}\right)^k > \frac{K}{S(0)u^n} \quad \Leftrightarrow \quad k < \frac{\ln(K/S(0)) - n\ln(u)}{\ln(d/u)}.$$

Taking into account the given data, the above inequality becomes $k < 2.11$. So, only the highest three stock prices yield a \$1 payoff. We need to calculate the risk-neutral probability of the stock price going up in a single step:

$$p^* = \frac{e^{0.11/11} - 0.96}{1.04 - 0.96} = 0.6256.$$

Finally, the cash-call price is

$$V_{CC}(0) = e^{-0.11}[(p^*)^{11} + 11(p^*)^{10}(1-p^*) + 55(p^*)^9(1-p^*)^2] = 0.1404.$$