MULTIPLE CHOICE QUESTIONS

1. (5 pts) Let $S$ be a non-dividend paying stock with a current price equal to $S_0$. You know that in the one-period binomial tree for this stock, $S_u = 150$ and $S_d = 120$. An actuary calculates the volatility of $S$ based on the provided values and gets $\sigma \approx 0.1116$. Which of the models we covered in class did this actuary use to obtain the volatility?
   (a) The forward binomial tree.
   (b) The Cox-Ross-Rubinstein model.
   (c) The lognormal tree.
   (d) Any of the three models.
   (e) None of the three models.

Solution: (d)
The ratio $S_u/S_d = u/d$ is always $e^{2\sigma}$ for all of the models we covered. In this case,

$$\frac{u}{d} = 1.25 = e^{2\sigma} \Rightarrow \sigma = \frac{1}{2} \ln(1.25) \approx 0.1116.$$

TRUE/FALSE QUESTIONS

1. (2 pts) Consider two exchange options, one that allows you to exchange a share of asset $S$ for a share of asset $Q$, and another one that allows you to forfeit a share of asset $Q$ and obtain a share of asset $S$ in return. Then, the prepaid forward prices of the two assets are the same if and only if the two exchange options have the same price.

Solution: TRUE

2. (2 pts) Consider a gap option whose trigger price is equal to its strike price. Then, the premium for this option is the same as that for an ordinary option with the same strike, the same exercise date and the same underlying asset.

Solution: TRUE
3. (2 pts) Consider a European gap put option such that its trigger price exceeds its strike price. Then, the premium of this option is decreasing with respect to the trigger price.

Solution: TRUE

Let us look at the payoff of this option at time \( T \); I am adding the trigger price \( K_2 \) in the notation to emphasise that we are considering it to be the argument of the payoff function.

\[
V_{GP}(T, K_2) = (K_1 - S(T))I_{[S(T) < K_2]}.
\]

Since we are given that \( K_1 < K_2 \), the above payoff is negative for all the values of \( S(T) \) such that \( K_1 < S(T) < K_2 \). Keeping all else fixed, and increasing the value of \( K_2 \), we see that the above region becomes “wider-and-wider”. It is evident that for all else kept intact, i.e., temporarily fixing \( K_1, S(T), \) and \( T \), the function \( V_{GP} \) is decreasing as a function of \( K_2 \). We have to conclude that this effect is reflected in the initial premium as well.

4. (2 pts) Consider two European exchange options both with exercise date \( T \), one that allows you to exchange a share of asset \( S \) for a share of asset \( Q \), and another one that allows you to forfeit a share of asset \( Q \) and obtain a share of asset \( S \) in return.

On the other hand, consider the maximum option with the payoff

\[
V_{\text{max}}(T) = \max(S(T), Q(T)),
\]

and the minimum option with the payoff

\[
V_{\text{min}}(T) = \min(S(T), Q(T)).
\]

Then, in our usual notation,

\[
V_{EC}(S(0), Q(0), 0) + V_{EC}(Q(0), S(0), 0) = V_{\text{max}}(0) + V_{\text{min}}(0).
\]

Solution: FALSE

If \( S(T) \leq Q(T) \), the payoff of a long exchange option allowing you to give up a unit of \( Q \) and receive a unit of \( S \) is

\[
V_{EC}(S(T), Q(T), T) = (S(T) - Q(T))_+ = 0,
\]

i.e., the option goes unexercised. On the other hand, the payoff of a long exchange option allowing you to give up a unit of \( S \) and receive a unit of \( Q \) is

\[
V_{EC}(Q(T), S(T), T) = (Q(T) - S(T))_+ = Q(T) - S(T).
\]

So, the payoff of the portfolio whose price is on the left-hand side of (1) is simply \( Q(T) - S(T) \).

The payoff of the portfolio whose initial cost is on the right-hand side of (1) is always \( S(T) + Q(T) \).

So, it is impossible for the proposed equality in prices to always be true.

5. (2 points) Let the continuously compounded interest rate be denoted by \( r \). Consider a futures contract for delivery at time \( -T \) of a market index with the continuous dividend yield \( \delta \). As a function of time, the price of this contract at time \( t \) is denoted by \( F_{t,T} \). Denote the time-\( t \) price of a European call on the futures contract with strike \( K \) and exercise date \( T^* < T \) by \( V_C(t) \), and denote the time-\( t \) price of a European put on the same futures contract with the same strike price and the same exercise date by \( V_P(t) \). Then, the following equality is always true

\[
V_C(t) - V_P(t) = F_{t,T}e^{-\delta(T-t)} - Ke^{-rT}.
\]
Solution: FALSE
There are many things amiss with the right-hand side of the above expression. The correct put-call parity for options on futures reads as

\[ V_C(t) - V_P(t) = e^{-r(T^*-t)}(F_{t,T} - K). \]

6. (2 points) In our usual notation, let \( S(0) = 40, r = 0.08, \sigma = 0.3, \delta = 0. \) You need to construct a 2-period forward binomial tree for the above stock with every period in the tree of length \( h = 0.5. \) Then, \( u > 1.45. \)
Solution: FALSE

\[ u = \exp\{(0.08 - 0) \cdot 0.5 + 0.3\sqrt{0.5}\} \approx 1.29. \]

7. (2 points) In the usual notation for the binomial asset pricing model, we always have

\[ 1 < d < u. \]

Solution: FALSE

8. (2 pts) In the setting of the binomial asset pricing model, with \( i \) denoting the effective interest rate per period and assuming that the underlying asset pays no dividends:

If

\[ d < u \leq 1 + i \]

then there is no possibility for arbitrage.
Solution: FALSE

9. (2 pts) Suppose that the European options with the same maturity and the same underlying assets have the following prices:

(1) a 50–strike call costs $9;
(2) a 55–strike call costs $10;

Then, some of the monotonicity conditions for no-arbitrage are violated by the above premiums.
Solution: TRUE

We know that for strikes \( K_1 < K_2, \) the price of a call with strike \( K_1 \) should be greater than or equal to the price of a call with strike \( K_2 \) (see equation (9.13) in the book). This condition is violated.

10. (2 pts) In the Cox-Ross-Rubinstein tree, we always have \( u = 1/d. \)
Solution: TRUE
FREE RESPONSE PROBLEMS

1. (20 points) The current price of a share of stock $S$ is $100. The stock is assumed to be paying a continuous dividend with the dividend yield of 0.04.

   Assume that the continuously compounded interest rate equals 0.05.

   Consider the following European gap options with the same exercise date in one year and the same underlying asset $S$.
   - I Gap call with strike price 100 and trigger price 100
   - II Gap put with strike price 100 and trigger price 100
   - III Gap call with strike price 100 and trigger price 110
   - IV Gap call with strike price 110 and trigger price 100
   - V Gap call with strike price 100 and trigger price 80.

   Which one of the above options has the highest price?

   Solution: Let us try compare the prices of options I and II, first. Since for the both of them the trigger and the strike prices are the same, we are in fact dealing with just plain vanilla options. The “regular” put-call parity applies, and in our usual notation, we have
   \[ V_I(0) - V_{II}(0) = F_{0,T}(S) - 100e^{-rT} = 100e^{-0.04} - 100e^{-0.05} = 100(e^{-0.04} - e^{-0.05}) > 0 \]

   Option III has a lower price than option I since the payoff curve for option I dominates the payoff of option III.

   Using the same type of comparison, we see that the value of option I is greater than the value of option IV (again, the payoff curve for option I is always above or at the same level as the payoff curve for option IV).

   Option I has the higher price than option V (again, its payoff curve is always above or at the same level as the payoff curve for option V). So, the price for option I is higher than the price of option V.

   We conclude that the option with the highest price of the ones offered is option I.

2. (20 points) Consider a one-period binomial model with $S(0) = 105$, $S_u = 130$ and $S_d = 80$. Your goal is to determine if there is an arbitrage opportunity in a market in which a European call option on $S$ with strike of $K = 120$ and exercise date $T = 1$ year is sold for $5. Assume that the continuously compounded risk-free interest rate equals $r = 10\%$.

   If you believe that there is an arbitrage opportunity, describe the arbitrage portfolio and show that it is, indeed, an arbitrage portfolio. If you believe that there is no arbitrage opportunity, explain your reasoning.

   Solution: First, we need to figure out the no-arbitrage price of the given call option. Since no dividends are mentioned, we set $\delta = 0$. The risk-neutral probability of the stock price going up is
   \[ p^* = \frac{e^{(r-\delta)h} - d}{u - d} = \frac{e^{0.1} - (80/105)}{(130 - 80)/105} = 0.72. \]
So, using the risk-neutral pricing formula, we get

$$V_C(0) = e^{-0.1} \cdot 0.72(130 - 120) = 6.51.$$ 

The observed price of the call is given to be $C = 5$. We conclude that there is an arbitrage opportunity since the no-arbitrage price (i.e., the value at time $0$ of the replicating portfolio for the call) is different from the observed call premium. Moreover, since the observed premium is lower than the no-arbitrage price, we conclude that the observed call is “relatively cheap” when compared to its replicating portfolio. To take advantage of this situation, we construct the following arbitrage portfolio:

1. **long** call,
2. **short** $\Delta = \frac{V_u - V_d}{S_u - S_d} = 0.2$ shares of stock,
3. **invest** $B = V_C(0) - \Delta S(0) = 6.51 - 0.2 \cdot 105 = -14.49$ in the money market at the risk-free interest rate $r$. **Note:** Since $B$ is a negative number, the interpretation is that we lend $|B|$ to the money market; this can be interpreted as purchasing zero-coupon bonds in the required amount to be redeemed at time $T$.

The initial cost of this portfolio is

$$C - \Delta S(0) + \left( -(V_C(0) - \Delta S(0)) \right) = C - V_C(0) < 0.$$ 

So, there is an initial inflow of money.

At the expiration date, the call and the short replicating portfolio have payoffs which exactly “cancel out” (by construction!).

Since there was an initial inflow of money, we indeed constructed an arbitrage portfolio. The arbitrage portfolio above is just an example of an arbitrage portfolio – it is by no means unique!

3. (10 points) Let $S(0) = 40, r = 0.08, \sigma = 0.3, \delta = 0$. You need to find the up and down factors in a 2–period forward binomial tree modeling the price of this stock during the following year.
   (a) (4 pts) What are $u$ and $d$?
   (b) (6 pts) What is the risk-neutral probability of the stock price going up in a single period?

**Solution:**

(a) 

$$u = e^{(r-\delta)\Delta + \sigma \sqrt{\Delta}} = e^{0.08 \cdot 0.5 + 0.3 \cdot \sqrt{0.5}} = 1.29;$$

$$d = e^{(r-\delta)\Delta - \sigma \sqrt{\Delta}} = e^{0.08 \cdot 0.5 - 0.3 \cdot \sqrt{0.5}} = 0.84.$$ 

(b) 

$$p^* = \frac{1}{1 + e^{\sigma \sqrt{\Delta}}} = 0.447.$$ 

4. (10 points)
   (i) (5 pts) Calculate the price of a long butterfly spread using the following call options:
      (1) a £3,925–strike call on the FTSE100 index which is being sold for £713.07;
      (2) a £4,325–strike call on the FTSE100 index which is being sold for £496.46;
      (3) a £4,725–strike call on the FTSE100 index which is being sold for £333.96.
(ii) (5 pts) Assume that the index pays no dividends. Use the put-call parity to derive the price of the corresponding butterfly spread in terms of the prices of put options analogous to the call options listed above.

**Solution:**

(i) The only thing one needs to remember here is how to construct a butterfly spread using calls. The butterfly spread above is symmetric. Its total cost can be evaluated as

\[ 713.07 - 2 \cdot 496.46 + 333.96 = 54.11 \]

(ii) First, we recall the general form of the put-call parity:

\[ V_C(t, K) - V_P(t, K) = F_{t,T}^P(S) - PV_{t,T}(K), \]

where \( V_C(t, K) \) denotes the time-\( t \) value of a European call with strike \( K \) and \( V_P(t, K) \) denotes the time-\( t \) value of a European put with strike \( K \). The price of the butterfly spread can be rewritten as

\[
\begin{align*}
V_C(t, 3925) - 2 \cdot V_C(t, 4325) + V_C(t, 4725) &= V_P(t, 3925) + F_{t,T}^P(S) - PV_{t,T}[3925] \\
&\quad - 2 \cdot (V_P(t, 4325) + F_{t,T}^P(S) - PV_{t,T}[4325]) + V_P(t, 4725) + F_{t,T}^P(S) - PV_{t,T}[4725] \\
&= V_P(t, 3925) - 2 \cdot V_P(t, 4325) + V_P(t, 4725) - PV_{t,T}[3295 - 2 \cdot 4325 + 4725] \\
&= V_P(t, 3925) - 2 \cdot V_P(t, 4325) + V_P(t, 4725) - PV_{t,T}[0].
\end{align*}
\]

As one would expect, the answer must be the same as in part (i), i.e., £54.11.

**Part IV. PROBLEMS WITH TABLES**

1. (18 points) Let \( S(0) = \$80, K = \$95, r = 8\%, T = 1 \) and \( \delta = 0 \). Assuming that \( u = 1.3 \) and \( d = 0.8 \), construct a two-period binomial tree for a call option. Provide all the entries in the following table:

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<th>Stock movement</th>
<th>Option premium</th>
<th>( \Delta )</th>
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**Solution: Note:** See Problem 10.5. from the textbook!
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