Discrete-dividend-paying stocks

Black-Scholes price

\[ V_C(0) = F_{0,T}^p(S) \cdot N(d_1) - F_{0,T}^p(K) \cdot N(d_2) \]

where

\[ d_1 = \frac{1}{\sigma \sqrt{T}} \left[ \ln \left( \frac{F_{0,T}^p(S)}{F_{0,T}^p(K)} \right) + \frac{\sigma^2 \cdot T}{2} \right] \]

and

\[ d_2 = d_1 - \sigma \sqrt{T} \]

\[ F_{0,T}^p(S) = S(0) - \sum_{k=1}^{n} PV_{0,t_k}(D_k) \]

\[ \text{dividend dates until the exercise date} \]
Problem

The current stock price is $7.20. The stock pays quarterly dividends of $0.10 w/ the next dividend scheduled for 1 month from now.

Assume $\Delta = 0.15$.

The continuously compounded risk-free interest rate is 0.06.

A European put expires in six months and has a strike price $K = 7.50$. Find the Black-Scholes price of this put!

→ \( 1^{st} \)

\[ F_{0,T}^P(S) = ? \]

\[ F_{0,T}^P(S) = 7.20 - 0.1 e^{-0.005} - 0.1 e^{-0.02} = 7.002 \]

\( 2^{nd} \)

\( \sigma = 0.15 \); \( K = 7.50 \)

\[ d_1 = \frac{1}{0.15 \sqrt{0.5}} \left[ \ln \left( \frac{7.002}{7.50} \right) + (r + \frac{\sigma^2}{2}) T \right] = -0.31 \]

\[ d_2 = -0.31 - 0.15 \cdot \sqrt{0.5} = -0.42 \]
\[ V^*_p(0) = Ke^{-rT}N(-d_2) - F_{o,T}^P(S) \cdot N(-d_4) \]
\[ V_p(0) = 7.50 e^{-0.03} \cdot \frac{N(0.42)}{0.6628} - 7.002 \cdot \frac{N(0.31)}{0.6217} \]
\[ V_p(0) = 0.472 \]

**Problem 15. [BLACK-SCHOLES FRAMEWORK]**

The current stock price is \( S(0) = 50 \).

The only dividend is \$1.50 to be paid in four months \( t_D = \frac{4}{3} \).

Consider a six-month European put w/ strike \( K = 50 \).

\( \sigma = 0.30 \)

Risk-free, continuously compounded \( r = 0.05 \).

\[
V_p(0) = K e^{-rT} N(-d_2) - F_{p,T}(S) N(-d_1)
\]

w/ \( d_1 = \frac{1}{\sigma \sqrt{T}} \left[ \ln \left( \frac{F_{p,T}(S)}{K} \right) + (r + \frac{1}{2} \sigma^2)T \right] \)

and \( d_2 = d_1 - \sigma \sqrt{T} \).

\[
F_{p,T}(S) = \frac{S(0)}{K} - 1.5 e^{-0.05 \cdot \frac{4}{3}} = 48.52
\]

\[
d_1 = \frac{1}{0.3 \sqrt{\frac{1}{2}}} \left[ \ln \left( \frac{48.52}{50} \right) + (0.05 + \frac{1}{2} \cdot 0.09) \cdot \frac{1}{2} \right] = 0.08
\]

\[
d_2 = 0.08 - 0.3 \cdot \sqrt{\frac{1}{2}} = -0.13 = 1 - N(0.08)
\]

\[
V_p(0) = 50 e^{-0.05 \cdot 0.5} N(0.13) - 48.52 \cdot N(-0.08)
\]

\[
V_p(0) = 4.20
\]
You are given:

\[ \frac{dS(t)}{S(t)} = 0.3 \, dt - \sigma \, dZ(t), \quad t \geq 0, \]

where \( Z(t) \) is a standard Brownian motion and \( \sigma \) is a positive constant.

(ii) There is a real number \( a \) such that

\[ \frac{d[S(t)]^a}{[S(t)]^a} = -0.66 \, dt + 0.6 \, dZ(t), \quad t \geq 0. \]

Calculate \( \sigma \).

A) 0.16
B) 0.20
C) 0.27
D) 0.60
E) 1.60

If \( S = \{ S(t), t \geq 0 \} \) is a GBm,

\[ \frac{dS(t)}{S(t)} = \mu_s \, dt + \sigma_s \, dZ(t) \]

then:

or explicitly

\[ S(t) = S(0) \, e^{(\mu_s - \frac{\sigma_s^2}{2}) \, t + \sigma_s \, Z(t)} \]

If you raise \( S \) to a power \( a \):

\[ (S(t))^a = (S(0))^a \, e^{a[(\mu_s - \frac{\sigma_s^2}{2}) \, t + \sigma_s \, Z(t)]} \]

\[ \Rightarrow S^a = \{(S(t))^a, t \geq 0\} \text{ is a GBm} \]
\[
\frac{dS(t)}{S(t)} = (\alpha - \delta)\,dt + \sigma\,dZ(t)
\]

Set: \( Y(t) = [S(t)]^a \) \( a \in \mathbb{R} \)

\[
dY(t) = ?
\]

Use Itô's lemma w/ \( F(t,x) = x^a \).

\[
\Rightarrow F_t = 0; \quad F_x(t,x) = a \cdot x^{a-1};
\]
\[
F_{xx}(t,x) = a(a-1)x^{a-2}.
\]

\[
\Rightarrow dY(t) = a \cdot [S(t)]^{a-1} \, dS(t) + \frac{1}{2} \cdot a(a-1) [S(t)]^{a-2} (dS(t))^2
\]
\[
dY(t) = a \cdot [S(t)]^{a-1} \left( S(t) \left( (\alpha - \delta)\,dt + \sigma\,dZ(t) \right) \right)
\]
\[
+ \frac{1}{2} a(a-1) [S(t)]^{a-2} \cdot [S(t)]^2 \sigma^2 dt
\]

\[
dY(t) = Y(t) \left[ a(\alpha - \delta) \,dt + a \cdot \sigma \,dZ(t) \right]
\]
\[
+ \frac{1}{2} a(a-1) \cdot \sigma^2 dt
\]

\[
\frac{dY(t)}{Y(t)} = \left( a(\alpha - \delta) + \frac{1}{2} a(a-1) \sigma^2 \right) dt + a \sigma \,dZ(t)
\]

THE VOLATILITY OF THE POWER OF RETURN
The dt-terms must match. So,
\[ a \cdot 0.3 + \frac{1}{2} a (a-1) \sigma^2 = -0.66 \]
The dZ(t)-terms also match. So,
\[ -a \cdot \sigma = 0.6 \Rightarrow \sigma = -\frac{0.6}{a} \]

\[ \Rightarrow 0.3a + 0.5 a(a-1) \left( -\frac{0.6}{a} \right)^2 = -0.66 \]
\[ 0.3a + 0.5a(a-1) \cdot \frac{0.36}{a^2} = -0.66 \quad \therefore a \]
\[ 0.3a^2 + 0.18(a-1) = -0.66a \]
\[ 0.3a^2 + 0.84a - 0.18 = 0 \quad \therefore 0.3 \]
\[ a^2 + 2.8a - 0.6 = 0 \]
\[ \begin{align*}
   a_1 &= -3 \\
   a_2 &= 0.2
\end{align*} \]
Keep this solution?

\[ \therefore \sigma = -\frac{0.6}{-3} = 0.2 \Rightarrow \Box \]