Exchange Option

T
S... underlying \( S_s, \sigma_s \)
A... strike asset \( S_a, \sigma_a \)
\( Z_s, Z_a \) "driving" sBms under the \( \mathbb{P}^* \) (risk-neutral probability measure)
\( \rho \)... the correlation coefficient between \( \hat{Z}_s \) and \( \hat{Z}_a \) (under \( \mathbb{P}^* \))

Black-Scholes

\[
V_{EC}(0) = F_{0,T}^p(S) \cdot N(d_1) - F_{0,T}^p(A) \cdot N(d_2)
\]

\( d_1 = \frac{1}{\sigma \sqrt{T}} \left[ \ln \left( \frac{F_{0,T}^p(S)}{F_{0,T}^p(A)} \right) + \frac{\sigma^2 T}{2} \right] \)

\[ d_2 = d_1 - \sigma \sqrt{T} \]

\[
\text{Var} \left[ \ln(S(t)) - \ln(A(t)) \right] = (\sigma_s^2 + \sigma_a^2 - 2 \rho \sigma_s \sigma_a) t = \sigma^2 t
\]

Another way of writing the pricing formula:

\[
F_{0,T}^p(S) = S(0) e^{-S_s T}
\]
\[
F_{0,T}^p(A) = A(0) e^{-S_a T}
\]

\[ d_1 = \frac{1}{\sigma \sqrt{T}} \left[ \ln \left( \frac{S(0)}{A(0)} \right) + (S_a - S_s + \frac{\sigma^2}{2}) T \right] \]
54. Assume the Black-Scholes framework. Consider two non-dividend-paying stocks whose time-\(t\) prices are denoted by \(S_1(t)\) and \(S_2(t)\), respectively.

You are given:

(i) \(S_1(0) = 10\) and \(S_2(0) = 20\).

(ii) Stock 1’s volatility is 0.18.

(iii) Stock 2’s volatility is 0.25.

(iv) The correlation between the continuously compounded returns of the two stocks is \(-0.40\).

(v) The continuously compounded risk-free interest rate is 5%.

(vi) A one-year European option with payoff \(\max\{\min[2S_1(1), S_2(1)] - 17, 0\}\) has a current (time-0) price of 1.632.

Consider a European option that gives its holder the right to sell either two shares of Stock 1 or one share of Stock 2 at a price of 17 one year from now.

Calculate the current (time-0) price of this option.

(A) 0.67
(B) 1.12
(C) 1.49
(D) 5.18
(E) 7.86
The payoff of the special option we need to price is:

\[(17 - Y(1))^+\]  \((*)\).

\((SP)\)  

\(\Rightarrow\) This is a put option on \(Y(1)\) w/ strike 17.

\((sc)\)  

In part (vi) we're given that the option w/ payoff \((Y(1) - 17)^+\) is worth 1.632 @ time 0.

\(\Rightarrow\) This is the "otherwise" identical call to the put from \((*)\).

\[\text{Put-call parity:} \quad V_{Sc}(0) - V_{Sp}(0) = \overline{F_{0,T}^P(Y)} - 17e^{-0.05}\]

\[1.632\]  

\[???\]

\[Y(1) = \min(2S_1(1), S_2(1)) \ldots \text{minimum option}\]
\[F_{0,1}^P(Y) = V_{\text{MIN}}(0)\]

\[\min(2S_1(1), S_2(1)) = 2S_1(1) + \min(0, S_2(1) - 2S_1(1))\]
\[= 2S_1(1) - \max(0, 2S_1(1) - S_2(1))\]
\[= 2S_1(1) - \overline{(2S_1(1) - S_2(1))^+}\]
\[(2S_1(n) - S_2(n))^+ \quad \text{payoff of an exchange call w/ underlying } 2S_1 \]
and strike asset \( S_2 \)

\[
\frac{dS(t)}{S(t)} = (\alpha - \delta) dt + \sigma \, dZ(t) \\
\text{Set: } \tilde{S}(t) = k \cdot S(t) \quad \text{for a constant } k \\
\Rightarrow \tilde{F}(t,x) = k \cdot x \\
\Rightarrow \tilde{F}_t = 0, \quad \tilde{F}_x = k, \quad \tilde{F}_{xx} = 0 \\
\text{ITO's Lemma:} \\
\frac{d\tilde{S}(t)}{\tilde{S}(t)} = k \, dS(t) \\
= \frac{k \cdot S(t) \, [(\alpha - \delta) dt + \sigma \, dZ(t)]}{\tilde{S}(t)} \\
\Rightarrow \frac{d\tilde{S}(t)}{\tilde{S}(t)} = (\alpha - \delta) dt + \sigma \, dZ(t) \\
\Rightarrow \text{In our problem, the volatility term of } 2S_1 \text{ is equal to the volatility of } S.\]
The time-0 price of our exchange call (note \( K \) is also preserved):

\[ V_{EC}(0) = 2F_{0,t_1}^p(S_1) \cdot N(d_1) - F_{0,t_1}^p(S_2) \cdot N(d_2) \]

\[ d_1 = \frac{1}{\sigma \sqrt{t}} \left[ \ln \left( \frac{2S_1(t)}{S_2(t)} \right) + (S_2 - S_1 + \frac{\sigma^2}{2}) \cdot 1 \right] \]

and \( d_2 = d_1 - \sigma \sqrt{t} \)

Where \( \sigma^2 = \sigma_1^2 + \sigma_2^2 - 2(-0.4) \cdot \sigma_1 \cdot \sigma_2 = \]

\[ = (0.18)^2 + (0.25)^2 + 0.8 \cdot 0.18 \cdot 0.25 \]

\[ \sigma = 0.3618 \]

\[ d_1 = \frac{1}{0.3618} \left( 0 - 0 + \frac{(0.3618)^2}{2} \right) = \frac{0.3618}{2} \]

\[ d_1 = 0.1809 \quad \Rightarrow \quad d_2 = -0.1809 \]

\[ V_{EC}(0) = 20 \left( N(0.1809) - N(-0.1809) \right) \cdot \frac{1 - N(0.1809)}{0.5718} \]

\[ \Rightarrow \quad V_{EC}(0) = 20 \left( 2N(0.1809) - 1 \right) = 2.87 \]

Answer: \( 1.632 - V_{Min}(0) + 17e^{-0.05} = 1.632 - 20 + 2.87 + 17e^{-0.05} = 0.673 \)
Option Greeks.

Consider a vanilla European call on a stock with strike $K$ and exercise $T$.

Black-Scholes price: @ time $-t$:

$$c(s, t, S, r, K, \sigma) = S(t) e^{-(T-t)\sigma^2} N(d_1) - Ke^{-(T-t)r} N(d_2)$$

$\sigma$ ... stock price on valuation date

with

$$d_1 = \frac{1}{\sigma\sqrt{T-t}} \left[ \ln \left( \frac{S(t)}{K} \right) + (r - \delta + \frac{\sigma^2}{2})(T-t) \right]$$

$$d_2 = d_1 - \sigma\sqrt{T-t}$$