On Market-Making and Delta-Hedging

1. Market Makers

2. Market-Making and Bond-Pricing
What to market makers do?

- Provide **immediacy** by standing ready to sell to buyers (at ask price) and to buy from sellers (at bid price)
- Generate **inventory** as needed by short-selling
- **Profit** by charging the bid-ask spread
- Their position is **determined** by the order flow from customers
- In contrast, **proprietary trading** relies on an investment strategy to make a profit
Delta neutral

- Stock will have the value of its Delta equal to zero - we say it is delta neutral.
- The portfolio which contains the option, along with \( AC \) shares of the underlying stock

The replicating portfolio will always contain \( AC \) shares of the underlying stock.

\[
\frac{S_0}{C_0} = \forall
\]

Recall the meaning of Delta.

- The largest part of the risk comes from the price movements of asset \( S \) - which is reflected in the delta of the option, i.e., if \( C \) is the asset \( S \) is most sensitive to the changes in the value of \( S \).
- An option written on an underlying asset \( S \) is most sensitive to the

\( \forall \)
23. Consider a European call option on a nondividend-paying stock with exercise date \( T, T > 0 \). Let \( S(t) \) be the price of one share of the stock at time \( t, t \geq 0 \). For \( 0 \leq t \leq T \), let \( C(s, t) \) be the price of one unit of the call option at time \( t \), if the stock price is \( s \) at that time. You are given:

(i) \[ \frac{dS(t)}{S(t)} = 0.1 dt + \sigma dZ(t) \], where \( \sigma \) is a positive constant and \( \{Z(t)\} \) is a Brownian motion.

(ii) \[ \frac{dC(S(t), t)}{C(S(t), t)} = \gamma(S(t), t)dt + \sigma C(S(t), t)dZ(t), \quad 0 \leq t \leq T \]

(iii) \( C(S(0), 0) = 6 \).

(iv) At time \( t = 0 \), the cost of shares required to delta-hedge one unit of the call option is 9.

(v) The continuously compounded risk-free interest rate is 4%.

Determine \( \gamma(S(0), 0) \).

(A) 0.10
(B) 0.12
(C) 0.13
(D) 0.15
(E) 0.16

\[ \Delta \text{-hedge a call:} \]

\{ 
- call
- investment in the underlying asset; \# of shares \( N(t) \)
\}

THE ENTIRE PORTFOLIO IS \( \Delta \text{-NEUTRAL} \)

The worth of this portfolio, as a function of \( s \) is:

\[ C(s, t) + N(t) \cdot s \]

\[ \Delta \text{-neutral means: } \frac{\partial}{\partial s} C(s, t) + N(t) = 0 \]

\[ \Rightarrow N(t) = -\Delta C(t) \]

Short shares!
Part (iv) \implies \text{cost of } \Delta\text{-hedging one written call is } q = \Delta_c(0) \cdot S(0)

**Option elasticity:** \[ \Omega = \frac{\Delta \cdot S}{p} \]

↑ price

The call & the underlying asset are driven by the same std BM \implies their Sharpe ratios are equal:

\[
\frac{\alpha_s - r}{\sigma_s} = \frac{\gamma(s, t) - r}{\sigma_c(s, t)}
\]

\[
\text{constant: } \frac{\alpha_s - r}{\sigma_s} = \Omega_c = \frac{\Omega_c}{\sigma_c(s, t)}
\]

\[\gamma(s(0), 0) - r = \frac{\Omega_c}{0.04} (\alpha_s - r)\]

\[q = \frac{3}{2} \]

\[\frac{0.1 - 0.04}{0.04} = 0.13 \rightarrow C\]

\[\gamma(s(0), 0) = 0.04 + \frac{3}{2} (0.1 - 0.04) = 0.13 \rightarrow C\]
SAMPLE MFE

65. Assume the Black-Scholes framework.

You are given:

(i) \( S(t) \) is the time-\( t \) price of a stock, \( t \geq 0 \).

(ii) The stock pays dividends continuously at a rate proportional to its price.

(iii) Under the true probability measure, \( \ln(S(2)/S(1)) \) is a normal random variable with mean 0.10.

(iv) Under the risk-neutral probability measure, \( \ln(S(5)/S(3)) \) is a normal random variable with mean 0.06.

(v) The continuously compounded risk-free interest rate is 4%.

(vi) The time-0 price of a European put option on the stock is 10.

(vii) For delta-hedging at time 0 one unit of the put option with shares of the stock, the cost of stock shares is 20.

Calculate the absolute value of the time-0 continuously compounded expected rate of return on the put option.

(A) 4% \( \frac{S(t)}{S(0)} = S(0) e^{(\alpha - \delta - \frac{\sigma^2}{2})t + \sigma Z(t)} \)

(B) 5%

(C) 10% \( \ln \left[ \frac{S(t+h)}{S(t)} \right] = (\alpha - \delta - \frac{\sigma^2}{2})h + \sigma [Z(t+h) - Z(t)] \)

(D) 11% \( \ln \left( \frac{S(t+h)}{S(t)} \right) \)

(E) 18%

\[ \Rightarrow \mathbb{E} \left[ \ln \left( \frac{S(t+h)}{S(t)} \right) \right] = (\alpha - \delta - \frac{\sigma^2}{2})h \]

\[ \text{under } \mathbb{P}^* : \quad \mathbb{E}^* \left[ \ln \left( \frac{S(t+h)}{S(t)} \right) \right] = (r - \delta - \frac{\sigma^2}{2})h \]
(iii) \( (\alpha - \delta - \frac{\sigma^2}{2}) \cdot 1 = 0.1 \) \( \Rightarrow \alpha - (\delta + \frac{\sigma^2}{2}) = 0.1 \)
(iv) \( (r - \delta - \frac{\sigma^2}{2}) \cdot 2 = 0.06 \)
(v) \( 0.04 - (\delta + \frac{\sigma^2}{2}) = 0.03 \)
\( \Rightarrow \delta + \frac{\sigma^2}{2} = 0.01 \)
\( \alpha - r = 0.07 \)

\[ \gamma_p (0) - r = \Omega_p (\alpha - r) \]

\[ \Omega_p = \frac{\Delta_p (0) \cdot S (0)}{P (0)} \]

(vii) \( \frac{-20}{10} = -2 \)

\[ \gamma_p (0) = 0.04 + (-2) (0.07) = -0.1 \] \( \Rightarrow \) C.

Just for laughs:

\[ \gamma = 0.04 + 2 \cdot 0.07 = 0.18 \]

Exactly offered answer \( \Box \)
\[
\frac{dS(t)}{S(t)} = \alpha_s \, dt + \sigma_s \, dZ(t)
\]

\[
\frac{dV_p(t)}{V_p(t)} = \alpha_p(t) \, dt + \sigma_p(t) \, dZ(t)
\]

\[
\Omega \cdot \sigma_s
\]

- Positive $\Omega$ leads to positive correlation between the underlying and the option price.
- Negative $\Omega$ leads to negative correlation.

Introduce $Z_p = -Z_{stock}$. 
Taylor-like expansions

\[ W(t) \]... total worth of a portfolio at time-\( t \) in the market model \( W \):

\[ \begin{cases} 
\text{risky asset} &: w/S(t), t \geq 0 \text{ its price @ time-} t \\
\text{risk-free} &: \text{@ a constant, continuously compounded } \gamma 
\end{cases} \]

\[ \Rightarrow \] Derivative securities on \( S \) also available.

\[ W(t + dt, S(t) + dS(t)) \approx W(t, S(t)) + \frac{\partial}{\partial t} W(t, S(t)) dt + \frac{\partial}{\partial S} W(t, S(t)) dS(t) + \frac{1}{2} \frac{\partial^2}{\partial S^2} W(t, S(t))(dS(t))^2 + \text{smaller/negligible terms} \]

\[ W(t + dt, S(t) + dS(t)) \approx W(t, S(t)) + \Theta(t) dt + \Delta_w(t) dS(t) + \frac{1}{2} \Pi_w(t)(dS(t))^2 \]

Delta-Gamma-Theta Approximation.
If we exclude $\Theta$, we get:

$$W(t+dt, s(t) + ds(t)) = W(t, s(t)) + \Delta w(t) ds(t) + \frac{1}{2} \Gamma w(t) (ds(t))^2$$

*Delta-Gamma Approximation*
19. Assume that the Black-Scholes framework holds. The price of a non-dividend-paying stock is $30.00. The price of a put option on this stock is $4.00.

You are given:

(i) $\Delta = -0.28$
(ii) $\Gamma = 0.10$

Using the delta-gamma approximation, determine the price of the put option if the stock price changes to $31.50.

\[
V_p(S(t) + dS(t)) \approx V_p(S(t)) + \Delta_p dS(t) + \frac{1}{2} \Gamma_p (dS(t))^2
\]

(A) $3.40$
(B) $3.50$
(C) $3.60$
(D) $\boxed{3.70}$
(E) $3.80$

\[
= \Gamma + (-0.28) \cdot (1.50) + \frac{1}{2} \cdot 0.10 \cdot (1.50)^2 = 3.69
\]

**END OF EXAMINATION**
Market maker: wrote a call; 
Δ-hedges using a stock investment.

Q: What is the portfolio's value in terms of \( S(t) \) and \( t \)?

Q: What does our market maker do after a little bit of time (say, a day) has passed?