Black-Derman-Toy Review.

\[ \frac{1}{2} \frac{1}{2} \frac{1}{2} \ldots \frac{1}{2} \]

\[ R_k \text{...rate level parameter} \]

\[ \sigma_k \text{...volatility of effective interest rates} \]

Equivalently, you can specify a bond price tree, i.e., a tree where at every node you have the price of a bond which matures at the end of that period for $1 (zero-coupon).

E.g.,

\[ r_0 \left\langle \begin{array}{c} \frac{1}{2} \\underbrace{\begin{array}{c} r_u \\ \frac{1}{2} \end{array}} \\ \frac{1}{2} \\underbrace{\begin{array}{c} r_d \\ \frac{1}{2} \end{array}} \end{array} \right. \]

\[ \left(=\right) \frac{1}{1+r_0} = P_0 \]

\[ P_u = \frac{1}{1+r_u} \]

\[ P_d = \frac{1}{1+r_d} \]
e.g.,

\[ r_0 \leq r_u \leq r_d \leq r_{dd} \leq r_{uu} \]

\[ \Rightarrow \quad p_0 \leq p_u \leq p_d \leq p_{dd} \]

\[
P_{uu} = \frac{1}{1+r_{uu}}
\]

\[
P_{ud} = \frac{1}{1+r_{ud}}
\]

\[
P_{dd} = \frac{1}{1+r_{dd}}
\]

\[
\{\text{rd}_d = ?\}\} \quad \text{needed}
\]

\[
\sigma_2 = ? \quad \text{parameters}
\]

\[
\Rightarrow \quad r_{ud} = r_{dd} e^{2\sigma_2 \sqrt{h}}
\]

and

\[
r_{uu} = r_{ud} e^{2\sigma_2 \sqrt{h}} = r_{dd} e^{4\sigma_2 \sqrt{h}}
\]

Q: Given \( r_{uu} \) and \( r_{dd} \). What is \( r_{ud} \)?

Note

\[
\frac{r_{uu}}{r_{ud}} = e^{2\sigma_2 \sqrt{h}} = \frac{r_{ud}}{r_{dd}}
\]

\[
\Rightarrow \quad r_{ud} = r_{uu} \cdot r_{dd} \Rightarrow \quad r_{ud} = \sqrt{r_{uu} \cdot r_{dd}}
\]
Problem 2.2. Forward on a bond
Consider the following values of interest rates from an incomplete Black-Derman-Toy interest rate tree for the effective annual interest rates.

\[ r_u = 0.35, \quad r_{uu} = 0.4, \quad r_d = 0.25, \quad r_{dd} = 0.20. \]

\[ \Rightarrow r_{ud} = \sqrt{r_{uu} \cdot r_{dd}} = \sqrt{0.08} = 0.28 \]

Let \( F \) denote the forward price for delivery at time-2 of a zero-coupon bond redeemable at time-3 for $1000. Then,

(a) \( 0 \leq F < 757 \)
(b) \( 757 \leq F < 767 \)
(c) \( 767 \leq F < 857 \)
(d) \( 857 \leq F < 915 \)
(e) None of the above.

\[ \text{answer: } 1000 \times \frac{P(0,3)}{P(0,2)} \]

\[ P(0,2) = \frac{1}{1 + r_o} \times \frac{1}{2} \times \left[ \frac{1}{1.35} + \frac{1}{1.25} \right] = \ldots. \]

\[ P(0,3) = \frac{1}{1 + r_o} \times \frac{1}{4} \times \left[ \frac{1}{1.35} \left( \frac{1}{1.4} + \frac{1}{1.28} \right) + \frac{1}{1.25} \left( \frac{1}{1.28} + \frac{1}{1.2} \right) \right] = \ldots. \]

\[ = \text{answer: } \frac{1000 \times P(0,3)}{P(0,2)} = \frac{1}{2} \left[ \frac{1}{1.35} + \frac{1}{1.25} \right] = 778.67. \]

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Problem 2.3. A call on a bond
Using the above BDT tree, let us find the price of a two-year, at-the-money call option on a three-year, zero-coupon bond redeemable for $1.

From solution to Problem 2.1.

\[ r_0 = 0.04 \]
\[ r_u = 0.0568 \]
\[ r_d = 0.0483 \]
\[ r_{uu} = 0.0727 \]
\[ P_{uu} = 1.0727^{-1} = 0.9322 \]
\[ r_{ud} = 0.0595 \]
\[ P_{ud} = 1.0595^{-1} = 0.9438 \]
\[ r_{dd} = 0.0487 \]
\[ P_{dd} = 1.0487^{-1} = 0.9536 \]

Exercise date of the call
Maturity date of the bond

At-the-money: strike price \[ K = P(0,3) = (1+r_0(0,3))^{-3} \]

The tree was calibrated to the given spot rates:

\[ K = 0.8638 \]

The call is always in-the-money on the exercise date:

\[ \text{Price: } \frac{1}{4} \times \frac{1}{1+r_0} \times \left[ \frac{1}{1+r_u} \times \frac{1}{1+r_{uu}} (P_{uu} - K) + \ldots \right] \]

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