Problem 2.4. BDT caplet pricing
Consider the following values of interest rates from an incomplete Black-Derman-Toy interest rate tree for the effective annual interest rates.

\[ r_0 = 0.09, \quad r_u = 0.12, \quad r_{uu} = 0.15, \quad r_d = 0.08, \quad r_{ud} = 0.13. \]

(i) (2 points) Find the volatility \( \sigma_1 \) of the interest rates at time 1.
(ii) (3 points) Find the interest rate \( r_{dd} \) missing from the tree.
(iii) (5 points) Consider a 3-year caplet for the notional amount of $100 whose cap rate is given to be 11.5%. Calculate its price.

\[ r_u = r_d e^{2\sigma_1} \quad \Rightarrow \quad \sigma_1 = \frac{1}{2} \ln \left( \frac{r_u}{r_d} \right) \]

\[ \sigma_1 = \frac{1}{2} \ln \left( \frac{0.12}{0.08} \right) = \frac{1}{2} \ln \left( \frac{3}{2} \right) = \ldots = 0.2027 \]

\[ \frac{r_{uu}}{r_{ud}} = e^{2\sigma_2} = \frac{r_{ud}}{r_{dd}} \quad \Rightarrow \quad r_{dd} = \frac{r_{ud}^2}{r_{uu}} = \frac{(0.13)^2}{0.15} = 0.1127 \]

(iii) Payments:
\[ @ \text{ uu : } 100 \times (0.15 - 0.115) \]
\[ @ \text{ ud : } 100 \times (0.13 - 0.115) \]

The caplet price:

\[ 100 \times \frac{1}{1.09} \times \frac{1}{4} \left[ (0.15 - 0.115) \times \frac{1}{1.15} \times \frac{1}{1.12} + \\
+ (0.13 - 0.115) \times \frac{1}{1.13} \left( \frac{1}{1.12} + \frac{1}{1.08} \right) \right] = \]

\[ = \ldots = 1.177 \]

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Yield Volatility

... "volatility in year-1" of an \( m \)-year, zero-coupon bond.

\[ R_0 \]

\[ R_d \]

\[ R_u \]

\[ \frac{1}{2} \]

\[ \frac{1}{2} \]

\[ \text{yields @ the up and down nodes} \]

The yield volatility is:

\[ \sigma = \frac{1}{2} \ln \left( \frac{y_u}{y_d} \right) \]
Problem 2.5. Yield volatility
In a Black-Derman-Toy tree, the annual effective interest rates (in our usual notation) are given to be
\[ r_0, \quad r_u = 0.05, \quad r_{uu} = 0.06, \]
\[ r_d = 0.045, \quad r_{dd} = 0.04. \]

(i) (5 points) Calculate \( r_{du} \).

(ii) (15 points) Compute the "volatility in year 1" of the 3 year zero-coupon bond generated by the tree.
Caveat: This is not the volatility of the effective interest rates in the tree at any single year!

\[ r_{du} = \sqrt{r_{uu}\cdot r_{dd}} = \ldots = 0.049 \]

\[ P_u = ? \quad , \quad P_d = ? \]

\[ y_u \quad \downarrow \quad \downarrow \quad y_d \]

\[ x = \frac{1}{2} \ln \left( \frac{y_u}{y_d} \right) = \text{answer} \]

\[ \begin{cases} 0.05 \quad \downarrow \quad \downarrow \quad 0.045 \quad \downarrow \quad \downarrow \quad 0.04 \\ \rightarrow \quad P_u = \frac{1}{1.05} \times \frac{1}{2} \times \left( \frac{1}{1.06} + \frac{1}{1.049} \right) = \ldots = 0.9032 \\ \rightarrow \quad P_d = \frac{1}{1.045} \times \frac{1}{2} \times \left( \frac{1}{1.049} + \frac{1}{1.04} \right) = \ldots = 0.9162 \end{cases} \]

The two possible values of: \( P_t(1,3) = P[1,3, r(u)] \)

\[ \begin{cases} P_u = (1 + y_u)^2 \Rightarrow y_u = P_u^{-\frac{1}{2}} - 1 = 0.0522 \\ P_d = (1 + y_d)^2 \Rightarrow y_d = P_d^{-\frac{1}{2}} - 1 = 0.0447 \end{cases} \]

\[ \Rightarrow \quad x = \frac{1}{2} \ln \left( \frac{0.0522}{0.0447} \right) = \ldots = 0.0746 \]
29. The following is a Black-Derman-Toy binomial tree for effective annual interest rates.

\[
\begin{array}{c c c}
\text{Year 0} & \text{Year 1} & \text{Year 2} \\
5\% & 6\% \\
3\% & r_{sd} \\
r_0 & 2\% \\
\end{array}
\]

Compute the “volatility in year 1” of the 3-year zero-coupon bond generated by the tree.

(A) 14\%
(B) 18\%
(C) 22\%
(D) 26\%
(E) 30\%
Solution to (29)  

Answer: (D)

According to formula (25.45) on page 771 in McDonald (2013), the “volatility in year 1” of an $n$-year zero-coupon bond in a Black-Derman-Toy model is the number $\kappa$ such that

$$y(1, n, r_u) = y(1, n, r_d) e^{2\kappa},$$

where $y$, the yield to maturity, is defined by

$$P(1, n, r) = \left( \frac{1}{1 + y(1, n, r)} \right)^{n-1}.$$

Here, $n = 3$. To find $P(1, 3, r_u)$ and $P(1, 3, r_d)$, we use the method of backward induction.

$$P(2, 3, r_{uu}) = \frac{1}{1 + r_{uu}} = \frac{1}{1.06},$$

$$P(2, 3, r_{dd}) = \frac{1}{1 + r_{dd}} = \frac{1}{1.02},$$

$$P(2, 3, r_{du}) = \frac{1}{1 + r_{du}} = \frac{1}{1 + \sqrt{r_{uu} \times r_{dd}}} = \frac{1}{1.03464},$$

$$P(1, 3, r_u) = \frac{1}{1 + r_u} \left[ \frac{1}{2} P(2, 3, r_{uu}) + \frac{1}{2} P(2, 3, r_{ud}) \right] = 0.909483,$$

$$P(1, 3, r_d) = \frac{1}{1 + r_d} \left[ \frac{1}{2} P(2, 3, r_{ud}) + \frac{1}{2} P(2, 3, r_{dd}) \right] = 0.945102.$$

Hence,

$$e^{2\kappa} = \frac{y(1,3,r_u)}{y(1,3,r_d)} = \frac{[P(1,3,r_u)]^{-1/2} - 1}{[P(1,3,r_d)]^{-1/2} - 1} = \frac{0.048583}{0.028633} = 0.264348 \approx 26\%.$$
15. You are given the following incomplete Black-Derman-Toy interest rate tree model for the effective annual interest rates:

```
   16.8%
   /     \
17.2%   12.6%
   /     \
9%     13.5%  11%
```

Calculate the price of a year-4 caplet for the notional amount of $100. The cap rate is 10.5%.
**Solution to (15)**

First, let us fill in the three missing interest rates in the B-D-T binomial tree. In terms of
the notation in Figure 25.4 of McDonald (2013), the missing interest rates are \( r_{dd} \), \( r_{ddd} \), and
\( r_{uud} \). We can find these interest rates, because in each period, the interest rates in
different states are terms of a geometric progression.

\[
\frac{0.135}{r_{dd}} = \frac{0.172}{0.135} \implies r_{dd} = 10.6\%
\]

\[
\frac{r_{uud}}{0.11} = \frac{0.168}{r_{uud}} \implies r_{uud} = 13.6\%
\]

\[
\left( \frac{0.11}{r_{ddd}} \right)^2 = \frac{0.168}{0.11} \implies r_{ddd} = 8.9\%
\]

The payment of a year-4 caplet is made at year 4 (time 4), and we consider its discounted
value at year 3 (time 3). At year 3 (time 3), the binomial model has four nodes; at that
time, a year-4 caplet has one of four values:

\[
\frac{16.8 - 10.5}{1.168} = 5.394, \quad \frac{13.6 - 10.5}{1.136} = 2.729, \quad \frac{11 - 10.5}{1.11} = 0.450, \quad \text{and} \quad 0 \quad \text{because} \quad r_{ddd} = 8.9\%
\]

which is less than 10.5%.

For the Black-Derman-Toy model, the risk-neutral probability for an up move is \( \frac{1}{2} \).
We now calculate the caplet’s value in each of the three nodes at time 2:

\[
\frac{(5.394 + 2.729)/2}{1.172} = 3.4654, \quad \frac{(2.729 + 0.450)/2}{1.135} = 1.4004, \quad \frac{(0.450 + 0)/2}{1.106} = 0.2034.
\]

Then, we calculate the caplet’s value in each of the two nodes at time 1:

\[
\frac{(3.4654 + 1.4004)/2}{1.126} = 2.1607, \quad \frac{(1.40044 + 0.2034)/2}{1.093} = 0.7337.
\]

Finally, the time-0 price of the year-4 caplet is \( \frac{(2.1607 + 0.7337)/2}{1.09} = 1.3277 \).

**Alternative Solution:** The payoff of the year-4 caplet is made at year 4 (at time 4). In a
binomial lattice, there are 16 paths from time 0 to time 4.

For the \( uuuu \) path, the payoff is \( (16.8 - 10.5) \).
For the \( uuud \) path, the payoff is also \( (16.8 - 10.5) \).
For the \( uudu \) path, the payoff is \( (13.6 - 10.5) \).
Coupled stock-price & interest-rate model

Two states-of-the-world after 1 step:

**STOCK PRICE**

- \( S(0) = S_0 \)
  - \( Su \) (up)
  - \( Sd \) (down)

**INTEREST RATE**

- \( r_0 \)
  - \( ru \) (up)
  - \( rd \) (down)

Timeline:

- \( 0 \) to \( 1 \)
- \( 1 \) to \( 2 \)