39. A discrete-time model is used to model both the price of a nondividend-paying stock and the short-term (risk-free) interest rate. Each period is one year.

At time 0, the stock price is $S_0 = 100$ and the effective annual interest rate is $r_0 = 5\%$.

At time 1, there are only two states of the world, denoted by $u$ and $d$. The stock prices are $S_u = 110$ and $S_d = 95$. The effective annual interest rates are $r_u = 6\%$ and $r_d = 4\%$.

Let $C(K)$ be the price of a 2-year $K$-strike European call option on the stock. Let $P(K)$ be the price of a 2-year $K$-strike European put option on the stock.

Determine $P(108) - C(108)$.

(A) $-2.85$
(B) $-2.34$
(C) $-2.11$
(D) $-1.95$
(E) $-1.08$
\[ S(0) = 100 \quad \left\langle \begin{array}{c} u \quad r_u = 0.06 \\ d \quad r_d = 0.04 \end{array} \right. \quad S_u = 110 \quad S_d = 95 \]

\[ \text{Put-Call Parity:} \]
\[ C(K) - P(K) = F_{0,T}^P(S) - PV_{0,T}(K) \]

\[ P(0,2) = \frac{1}{1+r_o} \times \left[ P \frac{1}{1+r_u} + (1-p) \frac{1}{1+r_d} \right] \]

\[ \text{risk-neutral probability (??)} \]
\[ \text{Must be consistent w/ stock-price tree.} \]

\[ P = \frac{S(0)(1+r_o) - S_d}{S_u - S_d} = \frac{105-95}{110-95} = \frac{2}{3} \]

\[ \Rightarrow P(0,2) = \frac{1}{1.05} \left[ \frac{2}{3} \cdot \frac{1}{1.06} + \frac{1}{3} \cdot \frac{1}{1.04} \right] = 0.9042 \]

\[ \Rightarrow \text{answer:} \quad 108 \cdot 0.9042 - 100 = -2.34 \quad \Rightarrow \mathbb{B} \]
30. You are given the following market data for zero-coupon bonds with a maturity payoff of $100.

<table>
<thead>
<tr>
<th>Maturity (years)</th>
<th>Bond Price ($)</th>
<th>Volatility in Year 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>94.34</td>
<td>N/A</td>
</tr>
<tr>
<td>2</td>
<td>88.50</td>
<td>10%</td>
</tr>
</tbody>
</table>

A 2-period Black-Derman-Toy interest tree is calibrated using the data from above:

```
Year 0
   ↘
  r_0
   ↗
   r_u

   ↘
  r_d

Year 1
```

Calculate \( r_d \), the effective annual rate in year 1 in the “down” state.

(A) 5.94%
(B) 6.60%
(C) 7.00%
(D) 7.27%
(E) 7.33%
Solution to (30) Answer: (A)

\[
\begin{align*}
\text{Year 0} & \quad \text{Year 1} \\
\uparrow & \quad \downarrow \\
\begin{array}{c}
r_0 \\
\downarrow \quad \downarrow \\
r_u = r_0 e^{2\sigma_1} \\
r_d
\end{array} & \quad \begin{array}{c}
r_0 \\
\downarrow \quad \downarrow \\
r_d
\end{array}
\end{align*}
\]

In a BDT interest rate model, the risk-neutral probability of each “up” move is \( \frac{1}{2} \).

Because the “volatility in year 1” of the 2-year zero-coupon bond is 10%, we have

\[ \sigma_1 = 10\%. \]

This can be seen from simplifying the right-hand side of (24.51).

We are given \( P(0, 1) = 0.9434 \) and \( P(0, 2) = 0.8850 \), and they are related as follows:

\[
P(0, 2) = P(0, 1)\left[\frac{1}{2}P(1, 2, r_u) + \frac{1}{2}P(1, 2, r_d)\right]
\]

\[
= P(0, 1)\left[\frac{1}{2}\frac{1}{21 + r_u} + \frac{1}{2}\frac{1}{21 + r_d}\right]
\]

\[
= P(0, 1)\left[\frac{1}{2}\frac{1}{21 + r_0 e^{0.2}} + \frac{1}{2}\frac{1}{21 + r_d}\right].
\]

Thus,

\[
\frac{1}{1 + r_0 e^{0.2}} + \frac{1}{1 + r_d} = \frac{2 \times 0.8850}{0.9434} = 1.8762,
\]

or

\[
2 + r_d (1 + e^{0.2}) = 1.8762[1 + r_d (1 + e^{0.2}) + r_d^2 e^{0.2}],
\]

which is equivalent to

\[
1.8762 e^{0.2} r_d^2 + 0.8762 (1 + e^{0.2}) r_d - 0.1238 = 0.
\]

The solution set of the quadratic equation is \{0.0594, \ -0.9088\}. Hence,

\[
r_d \approx 5.94\%.
\]
You are given the following information about a Black-Derman-Toy binomial tree modeling the movements of effective annual interest rates:

(i) The length of each period is one year.
(ii) In the first year, \( r_0 = 9\% \).
(iii) In second year, \( r_u = 12.6\% \) and \( r_d = 9.3\% \)
(iv) In third year, \( r_{uu} = 17.2\% \) and \( r_{dd} = 10.6\% \). The value of \( r_{ud} \) is not provided.

Calculate the price of a 3-year interest-rate cap for notional amount 10,000 and cap rate 11.5%.

(A) 202
(B) 207
(C) 212
(D) 217
(E) 222
from which we obtain \( r_{na} = 0.135 \). With this value, we can price the cap using (3).

Instead of using (3), we now solve the problem directly. As in Figure 25.6 on page 773 of McDonald (2013), we first discount each cap payment to the beginning of the payment year. The following tree is for notional amount of 1 (and \( K = 0.115 \)).

Discounting and averaging the cashflows at time 2 back to time 1:

\[
\begin{align*}
0.011 & \quad \frac{1}{1.126} + \frac{1}{1.126} \left( \frac{0.057}{2} + \frac{0.02}{1.135} \right) \\
0 & \quad \frac{1}{1.093} + \frac{1}{1.093} \left( \frac{0.02}{2} + \frac{0}{1.096} \right)
\end{align*}
\]

Thus the time-0 price of the cap is

\[
\frac{1}{1.09} \left\{ \frac{1}{1.126} \left[ 0.011 + \frac{1}{2} \left( \frac{0.057}{1.172} + \frac{0.02}{1.135} \right) \right] + \frac{1}{1.093} \left[ 0 + \frac{1}{2} \left( \frac{0.02}{1.135} + \frac{0}{1.106} \right) \right] \right\} = \frac{1}{1.09} \left\{ \frac{1}{1.126} \times 0.044128 + \frac{1}{1.093} \times 0.008811 \right\} = 0.02167474.
\]

**Answer (D) is correct, because the notation amount is 10,000.**

**Remark:** The prices of the two caplets for notional amount 10,000 and \( K = 0.115 \) are 44.81 and 171.94. The sum of these two prices is 216.75.
An analyst has used the following information to create a Black-Derman-Toy binomial tree with length of the binomial period equal to 1:

<table>
<thead>
<tr>
<th>Maturity in Years</th>
<th>Yield to Maturity</th>
<th>Bond Price</th>
<th>Volatility in Year 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5.0%</td>
<td>0.9524</td>
<td>N/A</td>
</tr>
<tr>
<td>2</td>
<td>6.0%</td>
<td>0.8900</td>
<td>8%</td>
</tr>
<tr>
<td>3</td>
<td>8.0%</td>
<td>0.7938</td>
<td>X</td>
</tr>
</tbody>
</table>

where volatility refers to the volatility of the bond price one year from today. The evolution in the price of the three-year zero coupon bond is given by the model as:

\[
\begin{align*}
0.7938 & \quad 0.8133 \quad 0.8555 \\
\quad & \quad 0.8537 \quad 0.8942 \quad 0.9234
\end{align*}
\]

t=0 \quad t=1 \quad t=2

Calculate the value X of the volatility in year 1 as shown in the table above of the 3-year bond that was used in constructing the tree.

A. Less than 13.5%
B. At least 13.5%, but less than 14.5%
C. At least 14.5%, but less than 15.5%
D. At least 15.5%, but less than 16.5%
E. At least 16.5%

\[\text{ans} \approx 13.9\%\]
Problem 1.5. (20 points) A discrete-time model is used to model both the price of a non-dividend-paying stock and the short-term (risk-free) interest rate. Each period is one year.

The time—0 stock price is given to be $80 per share, while the one-year spot rate for zero-coupon bonds equals 0.04.

At time—1, there are two states of the world: sunny and cloudy. In the sunny state of the world, the stock price is $100 and the effective yearly interest rate is 0.03. In the cloudy state of the world, the stock price is $75 and the effective yearly interest rate is 0.05.

Consider a European put and a European call on the above stock whose strike is $90 and whose exercise date is at time—2. What is the difference between the two option prices?
Problem 1.5. (20 points) A discrete-time model is used to model both the price of a non-dividend-paying stock and the short-term (risk-free) interest rate. Each period is one year.

The time-0 stock price is given to be $80 per share, while the one-year spot rate for zero-coupon bonds equals 0.04.

At time-1, there are two states of the world: sunny and cloudy. In the sunny state of the world, the stock price is $100 and the effective yearly interest rate is 0.03. In the cloudy state of the world, the stock price is $75 and the effective yearly interest rate is 0.05.

Consider a European put and a European call on the above stock whose strike is $90 and whose exercise date is at time −2. What is the difference between the two option prices?

Solution: The difference between the two options’ prices is

\[ V_C(0) - V_P(0) = F^P_{0,2}(S) - K P(0, 2) = S(0) - K P(0, 2) \]

The risk-neutral probability of the sunny state of the world is

\[ p^* = \frac{80(1.04) - 75}{100 - 75} = 0.328. \]

So,

\[ P(0, 2) = \frac{1}{1.04} \left[ \frac{1}{1.03} \times 0.328 + \frac{1}{1.05} \times (1 - 0.328) \right] = 0.9216. \]

Finally, our answer is

\[ 80 - 90(0.9216) = -2.9425. \]