Subjective Probabilities.

Investor forms an opinion, i.e., creates a model for the distribution of the stock price at a certain time- \( T \), subjective, or "TRUE", or ACTUAL, or historical.

\( S(T) \) ... random variable denoting the final asset price

\( S(T) \sim \text{Model/Distribution} \)

Focus: Expected value of the stock price, i.e.,

\[ \mathbb{E}[S(T)] \]

Assume: Our investors invest in a certain portfolio if \( \mathbb{E}[\text{Profit of portfolio}] > 0 \) i.e., they demand to be compensated for risk
Problem 2.1. MFE Sample (Introductory) Problem #6.
The following relates to one share of XYZ stock:

- The current price is 100.
- The forward price for delivery in one year is 105.
- An investor who decides to long the forward contract denotes by $\mathbb{E}$ the expected stock price in one year.

Determine which of the following statements about $P$ is TRUE.

- (A) $P < 100$
- (B) $P = 100$
- (C) $100 < P < 105$
- (D) $P = 105$
- (E) $P > 105$

**Profit = Payoff = \( S(T) - F \)**

Rational investor $\Rightarrow \mathbb{E}[S(T)] > F = 105$

Problem 2.2. MFE Sample (Introductory) Problem #38.
The current price of a medical company's stock is 75. The expected value of the stock price in three years is 90 per share. The stock pays no dividends. You are also given:

- The risk-free interest rate is positive.
- There are no transaction costs.
- Investors require compensation for risk.

The price of a three-year forward on a share of this stock is $X$, and at this price an investor is willing to enter into the forward. Determine what can be concluded about $X$.

- (A) $X < 75$
- (B) $X = 75$
- (C) $75 < X < 90$
- (D) $X = 90$
- (E) $X > 90$

\[ X < 90 \text{ same as Problem 2.1.} \]

\[ \text{also: } X = F_{0,3} (S) = 75 e^{3r} > 75 \]

\[ \uparrow r > 0 \]
Problem 2.3. MFE Sample (Introductory) Problem #70.

Investors in a certain stock demand to be compensated for risk. The current stock price is 100. The stock pays dividends at a rate proportional to its price. The dividend yield is 2%. The continuously compounded risk-free interest rate is 5%. Assume there are no transaction costs.

Let $X$ represent the expected value of the stock price 2 years from today. Assume it is known that $X$ is a whole number. Determine which of the following statements is true about $X$.

(A) The only possible value of $X$ is 105.
(B) The largest possible value of $X$ is 106.
(C) The smallest possible value of $X$ is 107.
(D) The largest possible value of $X$ is 110.
(E) The smallest possible value of $X$ is 111.

Profit: $e^{ST}S(T) - S(0)e^{rT}$
Demand: $E[\text{Profit}] > 0$

$\iff \quad X > E_{ST}(S) = 100e^{0.06 \times 106.18}$

Integers $\Rightarrow X \geq 107$

Problem 2.4. MFE Spring 2007: Problem #2

For a one-period binomial model for the price of a stock, you are given:

(i) The period is one year.
(ii) The stock pays no dividends.
(iii) $u = 1.433$, where $u$ is one plus the rate of capital gain on the stock if the price goes up.
(iv) $d = 0.756$, where $d$ is one plus the rate of capital loss on the stock if the price goes down.
(v) Calculate the true probability of the stock price going up.

The continuously compounded annual expected return on the stock is 10%.

(A) 0.52
(B) 0.57
(C) 0.62
(D) 0.67
(E) 0.72
One particular model is a stock-price binomial tree.

Stock w/ dividend yield S:

\[ S(0) \leftarrow S_u = u \cdot S(0) \]
\[ 1-p \quad S_d = d \cdot S(0) \]

\[ h \]

- Continuously compounded, risk-free i.r.
  \[ d < e^{(r-s)h} < u \]  \textbf{NO ARBITRAGE CONDITION!}

- Subjective probability of the stock price going up in a single step.

Say, you invest in one share of stock. Your expected wealth at the end of period = ?

\[ E[\text{Wealth}] = p \cdot e^{sh} \cdot u \cdot S(0) + (1-p) \cdot e^{sh} \cdot d \cdot S(0) \]
Let $\alpha$ satisfy:

$$S(0)e^{\alpha \cdot h} = E[\text{wealth}]
= p e^{s_h \cdot u \cdot S(0)} + (1-p) e^{s_h \cdot d \cdot S(0)}$$

We call $\alpha$ ... mean/expected rate of return

$$e^{\alpha \cdot h} = p e^{s_h \cdot u} + (1-p) e^{s_h \cdot d} \quad \therefore e^{s_h}$$

$$e^{(\alpha-s) \cdot h} = p \cdot u + (1-p) \cdot d$$

$$= p \cdot u + d - p \cdot d$$

$$\Rightarrow \quad p = \frac{e^{(\alpha-s) \cdot h} - d}{u - d}$$

**Note**: Under the risk-neutral probability, i.e., if we set $p = p^*$, we get $\alpha = r$. 
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\[
\begin{align*}
\text{Calculate!} \\
\frac{S_u}{S(0)} = \frac{S_u - S(0)}{S(0)} + 1 \\
\text{the rate of capital gain}
\end{align*}
\]

\[
\begin{align*}
\frac{S_d}{S(0)} = \frac{S_d - S(0)}{S(0)} + 1 \\
\text{the rate of capital loss}
\end{align*}
\]

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