Log-normal Stock Prices.

Review: $Y = e^X$ w/ $X \sim N(\text{mean} = m, \text{var} = \tau^2)$

- $\mathbb{E}[Y] = e^{m + \frac{\tau^2}{2}}$
  median of $Y : e^m$

$\Rightarrow$ median $\leq$ mean

- Compare: $\mathbb{E}[Y] = \mathbb{E}[e^X] = e^{m + \frac{\tau^2}{2}}$
  \(\text{vs.}\)

  $Y : e^{\mathbb{E}[X]} = e^m$

\[\text{exponential function}\]
Thm. [Jensen's Inequality]

If $X$ is a random variable, and $g$ is a **convex** function (assume all is well-defined below),

then:

$$E[g(X)] \geq g(E[X])$$

**e.g.** An example of a convex function is the payoff function for a call: $\psi_c(s) = (s-K)^+$

$$E[(g(T)-K)^+] \geq (E[S(T)]-K)^+$$
Proposed the log-normal stock prices:

\[ S(T) = S(0) e^{R(0,T)} \]

\[ R(0,T) \sim \text{Normal} \ (\text{mean} = \mu, \ \text{var} = \sigma^2) \]

Parameters for the stock price we want to incorporate in our model:

- \( \alpha \) .... mean rate of return
- \( \delta \) ... dividend yield
- \( \sigma \) .... volatility

In the market model, we also have:
- \( r \) ... continuously compounded risk-free i.r.

\[ \alpha - \delta \] ... rate of appreciation:

\[ S(0)e^{(\alpha - \delta)T} = \mathbb{E}[S(T)] \] by def'n.

\[ (LN) \Rightarrow \mathbb{E}[S(T)] = \mathbb{E}[S(0)e^{R(0,T)}] \]

\[ = S(0)e^{\mu + \frac{\sigma^2}{2}} \]

\[ \Rightarrow \text{The appropriate choice:} \]

\[ (\alpha - \delta)T = \mu + \frac{\sigma^2}{2} \] \( \Box \)
Q: What's the risk-neutral probability measure?

\[ \mathbb{E}^* [s(T)] = s(0) e^{(\alpha - \frac{\sigma^2}{2}) T + \sigma \sqrt{T} Z} \]

Q: What's the median time-T stock price?

\[ s(T) \]

By definition, \( \delta [R(0,T)] = 0 \)

\[ R(0,T) \sim N(\text{mean}=\alpha - \frac{\sigma^2}{2}) \]

\[ \text{var} = \sigma^2 T \]

\[ \frac{1}{2} \]
Problem. A non-dividend-paying stock is currently priced at $55 per share. The annual mean rate of return is 12%, and the annualized standard deviation of returns is 22%.

Model the stock price @ time-3 using the log-normal. What is the median time-3 stock price?

answer: 73.31.
Q: Given the $E[S(T)]$ and the median of $S(T)$, find the volatility $\eta$

$$E[S(T)] = \frac{S(0)e^{(d-S)T}}{S(0)e^{(d-S-\frac{\sigma^2}{2})T}} = e^{\frac{\sigma^2 T}{2}}$$

$$\eta = \frac{\text{median}}{S(0)e^{(d-S-\frac{\sigma^2}{2})T}}$$

$$\ln(\eta) = \frac{1}{2} \sigma^2 T \Rightarrow \frac{2\ln(\eta)}{T} = \sigma^2$$

$$\Rightarrow \sqrt{\frac{2\ln(\eta)}{T}} = \sigma$$

Q: Jarrow-Rudd tree: $u = e^{(r-S-\frac{\sigma^2}{2})\Delta t + \sigma \sqrt{\Delta t}}$

$\Rightarrow$ $d = e^{(r-S-\frac{\sigma^2}{2})\Delta t - \sigma \sqrt{\Delta t}}$

AKA: the lognormal tree