Lognormal Stock Prices [cont’d]

Review:
- $\alpha$ ... mean rate of return
- $S$ ... dividend yield
- $\sigma$ ... volatility.

Realized return: $[0, T]$

$$R(0, T) \sim N(\text{mean} = (\alpha - \delta - \frac{\sigma^2}{2})T, \text{var} = \sigma^2 T)$$

$\Rightarrow$ LogNormal stock prices:

$$S(T) = S(0) e^{R(0, T)}, \text{ i.e.,}$$

$w/ Z \sim N(0, 1)$, we have

$$S(T) = S(0) e^{(\alpha - \delta - \frac{\sigma^2}{2})T + \sigma \sqrt{T} Z}$$

Q: What is the expected wealth @ time $T$ of an investor who continuously reinvests the dividend in the same asset?

At $t = 0$:

\# of shares owned: 1 share $\rightarrow$ $e^{S_T}$ shares

wealth

$\Rightarrow$ the expected wealth equals:

$$\mathbb{E}[e^{S_T} S(T)] = e^{S_T} \mathbb{E}[S(T)] = \ldots$$
Problem.
A non-dividend-paying stock has the current price of $55 per share. Its mean rate of return is 12%, and its volatility is 0.22. Under the log-normal stock-price model, what is the median time-3 stock price?

\[
S(0) e^{(\alpha - \delta) T} = S(0) e^{0.08 \cdot 3} = 73.31.
\]

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Problem.
For a non-dividend-paying stock, we have:
• The mean rate of return is 10%.
• The median time-\(t\) stock price is
\[
S(0) e^{0.08 \cdot t} \quad \text{for all } t.
\]
What is the stock's volatility?

\[
m = (\alpha - \delta - \frac{\sigma^2}{2}) t \quad \text{with } \delta = 0, \alpha = 10\%
\]

\[
0.08t = (0.1 - \frac{0.2^2}{2}) t
\]

\[
\frac{0.2^2}{2} = 0.1 - 0.08 \quad \Rightarrow \quad \sigma = 0.2
\]
Tail probability

\[ P[S(T) > K] = ? \]

or

\[ P[S(T) < K] = ? \]

Focus on:

\[ P[S(T) > K] = P[S(0)e^{(\alpha-s-\frac{\sigma^2}{2})T + \sigma \sqrt{T} Z} > K] \]

with \( Z \sim N(0,1) \)

\[ P[S(T) > K] = P\left[(\alpha-s-\frac{\sigma^2}{2})T + \sigma \sqrt{T} Z > \ln\left(\frac{K}{S(0)}\right)\right] \]

\[ = P\left[\sigma \sqrt{T} Z > \ln\left(\frac{K}{S(0)}\right) - (\alpha-s-\frac{\sigma^2}{2})T\right] \]

\[ = P\left[Z > \frac{1}{\sigma \sqrt{T}} \left(\ln\left(\frac{K}{S(0)}\right) - (\alpha-s-\frac{\sigma^2}{2})T\right)\right] \]

\[ = P\left[Z < -\frac{1}{\sigma \sqrt{T}} \left(\ln\left(\frac{K}{S(0)}\right) - (\alpha-s-\frac{\sigma^2}{2})T\right)\right] \]

\[ = P\left[Z < -\frac{1}{\sigma \sqrt{T}} \left(\ln\left(\frac{S(0)}{K}\right) + (\alpha-s-\frac{\sigma^2}{2})T\right)\right] \]

\[ = \Phi\left(-\frac{1}{\sigma \sqrt{T}} \left(\ln\left(\frac{S(0)}{K}\right) + (\alpha-s-\frac{\sigma^2}{2})T\right)\right) \]
\[ P[S(T) > K] = N(\hat{d}_2) \]

\[ \Rightarrow P[S(T) < K] = N(-\hat{d}_2). \]

Q: What are the above probabilities under the risk-neutral probability measure \( \mathbb{P}^* \)?

\[ \mathbb{P}^* [S(T) > K] = \mathbb{P}^* [Z < \frac{1}{\sigma \sqrt{T}} (\ln(S(0)/K) + (r - \delta - \frac{\sigma^2}{2})T)] =: d_2 \]

\[ \mathbb{P}^* [S(T) > K] = N(d_2) \]

\[ \mathbb{P}^* [S(T) < K] = N(-d_2) \]

Example. Using the risk-neutral pricing principle, we get that:

- The price of a cash call is:
  \[ V_{CC}(0) = e^{-rT}N(d_2) \]

- The price of a cash put is:
  \[ V_{CP}(0) = e^{-rT}N(-d_2) \]