Problem 1.3. Forward on a bond
Consider the following values of interest rates from an incomplete Black-Derman-Toy interest rate tree for the effective annual interest rates.

\[ r_u = 0.35, \quad r_{uu} = 0.4, \]
\[ r_d = 0.25, \quad r_{dd} = 0.20. \]

Let \( F \) denote the forward price for delivery at time-2 of a zero-coupon bond redeemable at time-3 for $1000. Then,

(a) \( 0 \leq F < 757 \)
(b) \( 757 \leq F < 787 \)
(c) \( 787 \leq F < 857 \)
(d) \( 857 \leq F < 915 \)
(e) None of the above.

\[ F = F_{2,3} \left[p(2,3)\right] = \frac{p(0,3)}{p(0,2)}. \]

\[ r_{uu} = 0.4 \]
\[ r_{u} = 0.35 \]
\[ r_{d} = 0.25 \]
\[ r_{dd} = 0.2 \]
\[ r_{ud} = r_{du} = ? \]
\[ r_{ud} = r_{dd} \cdot e^{2\sigma^2} \]

\[ r_{ud}^2 = r_{dd} \cdot r_{uu} = 0.08 \]

\[ r_{ud} = 0.2828 \]

\[ P(0, 2) = \frac{1}{1+r_0} \times \frac{1}{2} \times \left[ \frac{1}{1+r_u} + \frac{1}{1+r_d} \right] \]

\[ P(0, 3) = \frac{1}{1+r_0} \times \frac{1}{4} \times \left[ \frac{1}{1+r_u} \times \frac{1}{1+r_{uu}} + \frac{1}{1+r_u} \times \frac{1}{1+r_{ud}} + \frac{1}{1+r_{u}} \times \frac{1}{1+r_{ud}} + \frac{1}{1+r_{ud}} \times \frac{1}{1+r_{dd}} \right] \]

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\[ F = \frac{1}{2} \times \left[ \frac{1}{1.35} \cdot \frac{1}{1.4} + \frac{1}{1.35} \cdot \frac{1}{1.28} + \frac{1}{1.25} \cdot \frac{1}{1.28} + \frac{1}{1.25} \cdot \frac{1}{1.2} \right] \]

\[ F = 777.8 \]
Problem 1.2. A call on a bond
Using the above BDT tree, let us find the price of a two-year at-the-money call option on a three-year, zero-coupon bond redeemable for $1.

\[ r_0(0,3) = 0.05 \]

\[ r_0 = 0.04 \]

\[ r_u = 0.0568 \quad r_d = 0.0483 \]

\[ r_{uu} = 0.0727 \quad r_{dd} = 0.0487 \]

\[ r_{du} = 0.0595 \]

\[ T_0 = 2 \quad T_M = 3 \]

The call's expiration date
The bond's maturity

\[ K = P(0,3) = \frac{1}{(1 + r_0(0,3))^3} = \frac{1}{1.05^3} = 0.8638 \]

\[ P_{uu} = \frac{1}{1 + r_{uu}} = \frac{1}{1.0727} = 0.9322 \]

\[ P_{ud} = \frac{1}{1 + r_{du}} = \frac{1}{1.0595} = 0.9438 \]

\[ P_{dd} = \frac{1}{1 + r_{dd}} = \frac{1}{1.0487} = 0.9536 \]
In general:

\[ V_c(0) = \frac{1}{1+r_o} \times \frac{1}{4} \left[ \frac{1}{1+r_u} (P_{uu} - K)_+ + (P_{ud} - K)_+ \\
+ \frac{1}{1+r_d} ((P_{ud} - K)_+ + (P_{dd} - K)_+) \right] \]

In this problem, the call is ALWAYS in-the-money at expiry. So,

\[ V_c(0) = \frac{1}{4(1+r_o)} \left[ \frac{1}{1+r_u} (P_{uu} + P_{ud}) + \frac{1}{1+r_d} (P_{ud} + P_{dd}) \\
- K \left( \frac{1}{1+r_u} + \frac{1}{1+r_d} \right) \cdot 2 \right] \]

\[ = P(0,3) - K \cdot P(0,2) \]

In particular, if @the-money @ time-0:

\[ V_c(0) = P(0,3) - P(0,3)(P(0,2)) = P(0,3)(1 - P(0,2)) \]

\[ = \frac{1}{1.05^3} (1 - \frac{1}{1.045^2}) = \ldots. \]

Spot rates
Problem 1.4. BDT caplet pricing
Consider the following values of interest rates from an incomplete Black-Derman-Toy interest rate tree for the effective annual interest rates.

\[ r_0 = 0.09, \quad r_u = 0.12, \quad r_{uu} = 0.15, \]
\[ r_d = 0.08, \quad r_{ud} = 0.13. \]

(i) (2 points) Find the volatility \( \sigma_1 \) of the interest rates at time \(-1\).
(ii) (3 points) Find the interest rate \( r_{dd} \) missing from the tree.
(iii) (5 points) Consider a 3–year caplet for the notional amount of $100 whose cap rate is given to be 11.5%. Calculate its price.

\[ r_0 = 0.09 \]
\[ r_u = 0.12 \]
\[ r_d = 0.08 \]
\[ r_{uu} = 0.15 \]
\[ r_{ud} = 0.13 \]

\[ \sigma_1 = ? \]
\[ r_u = r_d e^{2\sigma_1} \]
\[ \frac{r_u}{r_d} = e^{2\sigma_1} \]
\[ \sigma_1 = \frac{1}{2} \ln \left( \frac{r_u}{r_d} \right) = 0.2027 \]

\[ r_{dd} = \frac{r_{ud}^2}{r_{uu}} = \frac{0.13^2}{0.15} = \frac{0.0169}{0.15} = 0.1127 \]

\[ r_{ud} = r_{dd} \cdot e^{2\sigma_2} \]
\[ r_{uu} = r_{dd} \cdot e^{2\sigma_2} \]

\[ \Rightarrow \quad \frac{r_{uu}}{r_{ud}} = \frac{r_{ud}}{r_{dd}} \Rightarrow \frac{r_{ud}^2}{r_{uu}} \]
\[ r_{dd} = \frac{r_{ud}^2}{r_{uu}} \]