Note:
The following problems are meant to be the special HW for graduate students. If you are taking this course as an undergraduate course, your score on will count as extra credit. Details are below.
A part of the homework involves simulations while the rest are calculations involving the random walk. For the simulations part, you are allowed to use any software you are comfortable with. Examples of convenient software are: Mathematica, Matlab, Excel, R. Please, do not hand in a printout of your homework solutions. Send them to me as a .pdf attachment.
If you are a graduate student, this HW will be taken into account when your HW average is calculated (see the First-Day Handout). If you are taking this course as an undergraduate course, then your score in this assignment will serve as extra credit toward your final score in the course. More precisely, your score in this homework will be given the weight of 5% and incorporated in the final score in addition to the points earned through other assessments.
1.1. The Trinomial Model.

Problem 1.1. (5 points) Solve Sample MFE Problem #27.

Problem 1.2. (10 points) Consider the following coupled trinomial model for the prices in one year of a pair of non-dividend-paying stocks $S$ and $Q$. Assume that the continuously compounded risk-free interest rate equals 0.04.

Consider a one-year, $90$-strike European put option on $S$. What is the price of this option?
1.2. The Cointoss Space. For the following problems, please use
https://www.ma.utexas.edu/users/mcudina/m339w_ProbCoinToss.pdf
as a reference.

Problem 1.3. (14 points) Source: “Binomial Option Pricing” by Shreve.
Consider the stock price binomial tree defined by $S(0) = 4$, $u = 2$ and $d = 1/2$.

(i) (4 pts) What is the distribution of the random variable $S(3)$ under the risk-neutral
probabilities $p^* = q^* = 1/2$?

(ii) (4 pts) Compute $E^*[S(1)]$, $E^*[S(2)]$ and $E^*[S(3)]$. What is the average rate of growth
of the stock price under the risk-neutral probability measure $P^*$ given through the risk-
neutral probabilities $p^*$ and $q^*$?

(iii) (6 pts) Answer the questions in (i) and (ii) again, but this time under the actual
probabilities $p = 2/3$ and $q = 1/3$.

Problem 1.4. (14 points) Source: “Binomial Option Pricing” by Shreve.
An illustration of the Iterated Conditioning property of conditional expectation. Consider the
same binomial tree modeling the stock price $S$ as in the above problem. Recall the notation we
introduced in class and find the following, using the actual probabilities $p = 2/3$ and $q = 1/3$:

(i) (4 pts) the random variable $E_2[S(3)]$; i.e., find the values this random variable attains
at different elements of the coin toss sample space, i.e.,

\[ E_2[S(3)](HH), E_2[S(3)](HT), E_2[S(3)](TH), E_2[S(3)](TT); \]

(ii) (4 pts) based on your answers to part (i), the conditional expectation $E_1[E_2[S(3)]]$, i.e.,

\[ E_1[E_2[S(3)]|(H), E_1[E_2[S(3)]|(T); \]

(iii) (4 pts) the random variable $E_1[S(3)]$ - but only from first principles (not using the iterated
conditioning property); i.e., find the values this random variable attains at different
elements of the coin toss sample space, i.e.,

\[ E_1[S(3)](H), E_1[S(3)](T). \]

(iv) (2 pts) Explain in a sentence or two and using at most one formula how the above
calculations illustrate the iterated conditioning property of conditional expectation.
Problem 1.5. (8 points) Source: “Binomial Option Pricing” by Shreve.
Let the dynamic of the price of a certain stock be modeled using the binomial tree. Under this assumption, the random variables denoting the possible prices of the stock at times 0, 1, 2, 3 are given as follows:

\[
\begin{align*}
S(0)(\omega_1\omega_2\omega_3) &= 4 \text{ for all } \omega_1\omega_2\omega_3 \in \Omega \\
S(1)(\omega_1\omega_2\omega_3) &= \begin{cases} 
8 & \text{for } \omega_1 = H \\
2 & \text{for } \omega_1 = T 
\end{cases} \\
S(2)(\omega_1\omega_2\omega_3) &= \begin{cases} 
16 & \text{for } \omega_1 = \omega_2 = H \\
4 & \text{for } \omega_1 \neq \omega_2 \\
1 & \text{for } \omega_1 = \omega_2 = T 
\end{cases} \\
S(3)(\omega_1\omega_2\omega_3) &= \begin{cases} 
32 & \text{for } \omega_1 = \omega_2 = \omega_3 = H \\
8 & \text{if there are two heads and one tail} \\
2 & \text{if there are two tails and one head} \\
0.5 & \text{for } \omega_1 = \omega_2 = \omega_3 = T
\end{cases}
\]

In this problem you are going to verify the “taking out what is known” property of conditional expectation using the model outlined above. Assume that the probability of the stock price moving UP at any of the nodes equals \( p = \frac{2}{3} \).

(i) (2 pts) Calculate \( \mathbb{E}_1[S(2)] \), i.e., find \( \mathbb{E}_1[S(2)(H)] \) and \( \mathbb{E}_1[S(2)(T)] \).
(ii) (4 pts) Calculate \( \mathbb{E}_1[S(1) \cdot S(2)] \), i.e., find \( \mathbb{E}_1[S(1) \cdot S(2)(H)] \) and \( \mathbb{E}_1[S(1) \cdot S(2)(T)] \).
(iii) (2 points) Explain with one or two sentences and a couple of formulas how the above computations illustrate the “taking-out-what-is-known” property of conditional expectation.
Problem 1.6. (16 points) Source: “Binomial Option Pricing" by Shreve.
Let the dynamic of the price of a certain stock be modeled using the binomial tree. Under this
assumption, the random variables denoting the possible prices of the stock at times 0, 1, 2, 3 are
given as follows:

\[ S(0)(\omega_1 \omega_2 \omega_3) = 4 \text{ for all } \omega_1 \omega_2 \omega_3 \in \Omega \]
\[ S(1)(\omega_1 \omega_2 \omega_3) = \begin{cases} 
8 & \text{for } \omega_1 = H \\
2 & \text{for } \omega_1 = T 
\end{cases} \]
\[ S(2)(\omega_1 \omega_2 \omega_3) = \begin{cases} 
16 & \text{for } \omega_1 = \omega_2 = H \\
4 & \text{for } \omega_1 \neq \omega_2 \\
1 & \text{for } \omega_1 = \omega_2 = T 
\end{cases} \]
\[ S(3)(\omega_1 \omega_2 \omega_3) = \begin{cases} 
32 & \text{for } \omega_1 = \omega_2 = \omega_3 = H \\
8 & \text{if there are two heads and one tail} \\
2 & \text{if there are two tails and one head} \\
0.5 & \text{for } \omega_1 = \omega_2 = \omega_3 = T 
\end{cases} \]

In this problem you are going to verify the linearity property of conditional expectation using
the model outlined above. Assume that the probability of the stock price moving UP at any of
the nodes equals \( p = 2/3 \).

(i) (2 pts) Draw the binomial tree corresponding to the above stock prices and enter all
numerical values for the stock prices at their appropriate nodes within the tree.

(ii) (2 pts) Calculate \( \mathbb{E}_1[S(2)] \), i.e., find \( \mathbb{E}_1[S(2)(H)] \) and \( \mathbb{E}_1[S(2)(T)] \).

(iii) (4 pts) Calculate \( \mathbb{E}_1[S(3)(H)] \), i.e., find \( \mathbb{E}_1[S(3)(H)] \) and \( \mathbb{E}_1[S(3)(T)] \).

(iv) (6 pts) Calculate from first principles (not using linearity) \( \mathbb{E}_1[S(2 + S(3)] \), i.e., find
\( \mathbb{E}_1[S(2 + S(3)(H)] \) and \( \mathbb{E}_1[S(2 + S(3)(T)] \).

(v) (2 points) Explain in one or two sentences and using a single formula how the above
computations illustrate the linearity property of conditional expectation.
1.3. A Simple Random Walk.

**Problem 1.7.** (20 points) Let \( \{X_n, n = 0, 1, 2, \ldots \} \) be a symmetric simple random walk. More precisely, let \( \xi_k, k = 1, 2, \ldots \) be a sequence of independent identically distributed random variables such that

\[
P[\xi_1 = 1] = P[\xi_1 = -1] = \frac{1}{2}.
\]

Then, \( \{X_n, n = 0, 1, 2, \ldots \} \) is defined as

\[
X_0 = 0, \quad X_n = \xi_1 + \xi_2 + \cdots + \xi_n.
\]

Compute the following

1. (5 points) \( P[X_{2n} = 0], n \in \mathbb{N}_0 \),
2. (5 points) \( P[X_n = X_{2n}], n \in \mathbb{N}_0 \),
3. (5 points) \( P[|X_1X_2X_3| = 2] \),
4. (5 points) \( P[X_7 + X_{12} = X_1 + X_{16}] \).

1.4. Scaled Random Walks. For the following problems please use

https://www.ma.utexas.edu/users/mcudina/m339w_scaled_rnd_walks.pdf as a reference.

**Problem 1.8.** (20 points) Using the terminology and the notation of the lectures on the scaled symmetric random walks, calculate the following values:

1. (3 pts) \( E[W^{(100)}(0.70) - W^{(100)}(0.30)] \)
2. (3 pts) \( Var[W^{(100)}(0.60) - W^{(100)}(0.30)] \)
3. (3 pts) \( [W^{(100)}, W^{(100)}](1.35) \)
4. (3 pts) \( P[W^{(100)}(0.15) = 0.1] \)
5. (8 pts) \( \lim_{n \to \infty} E[\min(5, (W^{(n)}(7) - 5)^+)] \)

**Problem 1.9.** (12 points)

1. (5 pts) Using the terminology and the notation of the lectures on the scaled symmetric random walks, simulate 1000 values of \( W^{(100)}(0.15) \) and draw the histogram of the obtained set of values.
2. (2 pts) Find the median of the set of simulated values you obtained in part (i).
3. (2 pts) Find the average of the set of simulated values you obtained in part (i).
4. (3 pts) Draw the graph of the density of a normal distribution with the appropriate mean and variance that, together with the histogram from part (i), illustrates the Central Limit Theorem.
1.5. The Inverse Transformation Method.

Problem 1.10. (8 pts) Using the inverse-transformation method, simulate 100 values from the exponential distribution with the (mean) parameter $\lambda$ equal to 2. Draw a histogram of the simulated values. Show your code!